Solutions 13.

13-1: Assume $k$ coordinates of $x$ are 1 and $n - k$ are 0. Then, $v \in \mathbb{F}_2^n$ is orthogonal to $x$ if and only if it has an even number of ones at those $k$ coordinates. The number of even subsets of a $k$-element set is \( \binom{k}{0} + \binom{k}{2} + \binom{k}{4} + \ldots + \binom{k}{\lfloor k/2 \rfloor} = 2^{k-1} \), as we have learned. The other $n - k$ coordinates may be chosen freely. Thus, the number of orthogonal vectors is $2^{k-1} \cdot 2^{n-k} = 2^{n-1}$, which is half the elements of $\mathbb{F}_2^n$.

13-2: We prove it for even $n$, as for odd $n$ the statement follows from the even case. We make $n/2$ pairs, say $A := \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{n-1, n\}\}$. Now, for any subset $B$ of $A$, we take the union of the members of $B$. The collection of these unions is $\mathcal{F}$. It clearly satisfies the conditions.

13-3: Assume that $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$ for some real numbers $\alpha_1, \ldots, \alpha_n$. We need to show that $\alpha_1 = \ldots = \alpha_n = 0$.

Take the scalar product of the two sides of the first equation with $v_1$. Then,

$$\alpha_1 v_1 \cdot v_1 + \ldots + \alpha_n v_n \cdot v_1 = 0 \cdot v_1 = 0.$$

Since the vectors are pairwise orthogonal, on the right hand side, we get $\alpha_1 v_1 \cdot v_1 = 0$. Since $v_1 \neq 0$, we have $v_1 \cdot v_1 > 0$, and thus, $\alpha_1 = 0$. In the same way, we obtain that the other coefficients are also zero.

13-4: We prove by induction on $n$. Note that since we are working over $\mathbb{F}_2$, all exponents are 0 or 1. Since $f$ is not identically 1, at least one of the variables is present in $f$, we may assume that $x_n$ is such. Now, some terms contain $x_1$, some do not. Thus, $f$ can be written as

$$f(x_1, \ldots, x_n) = x_n g(x_1, \ldots, x_{n-1}) + h(x_1, \ldots, x_{n-1}),$$

where $g(x_1, \ldots, x_{n-1})$ is a not identically 0 polynomial of $n - 1$ variables and of degree at most $d - 1$. By the induction hypotheses, there is an assignment of values of $x_1, \ldots, x_{n-1}$, with at most $d$ of them 1, such that $g(x_1, \ldots, x_{n-1}) + 1$ is 0. Thus, for this assignment, $g(x_1, \ldots, x_{n-1}) = 1$. Now, if $h(x_1, \ldots, x_{n-1}) = 1$ then we set $x_n = 1$, otherwise we set $x_n = 0$. 