Problem Set 11.

11-1: Find the generating function of the following sequences
(a) \((1, 2, 4, 8, \ldots)\), that is, \(a_n = 2^n\);
(b) \((1, 0, 2, 0, 4, 0, 8, \ldots)\), that is, \(a_n = 2^{n/2}\) for even \(n\) and \(a_n = 0\) for odd \(n\);
(c) \((1, 1, 2, 4, 8, 8, \ldots)\), that is, \(a_n = 2^\lfloor n/2 \rfloor\).

11-2:
(a) Write the Taylor series expansion around \(x = 0\) of \(a''(x)\), where \(a(x) = \frac{1}{1-x}\).
(b) What is the generating function for the sequence \((1^2, 2^2, 3^2, \ldots)\) of squares, that is, for the sequence \(a_n = (n + 1)^2\)?

11-3: Find the generating function of the sequence \((a_0, a_1, \ldots)\) with the following recursive definition: \(a_0 = 0, a_1 = 1\) and \(a_{n+2} = 2a_{n+1} - a_n\).

11-4: Write the recurrence relation for the sequence \((a_0, a_1, \ldots)\) where \(a_n\) is the number of ways we can climb \(n\) stairs so that in each step we climb 1 or 3 stairs. Write the generating function for this sequence.

11-5: Prove that any positive integer can be written as a sum of mutually distinct Fibonacci numbers.

11-6: * One thousand and one members of parliament sit around a round table. Each of them either supports Brexit or supports Remain. Every day each of them announces her opinion loudly. A member of parliament changes her opinion after the announcements are made if and only if both of her neighbors’ opinions are the opposite of hers. Prove that after a while, there will be no changes of opinion.