Solutions 9.

9-1:
(a) The probability of picking a girl = 9/21 = 3/7.
(b) The probability of picking a girl, provided that we pick a 3-year-old = 4/7.
(c) The probability of picking a 3-year-old, provided that it is a girl = 4/9.

9-2:
(a) We throw a fair die. Let $X$ be the value mod 5, that is, $X$ is 1,2,3,4,0,1 according to the die values 1,2,3,4,5,6. Let $Y$ be the value of the die roll mod 2, that is, $Y$ is 1,0,1,0,1,0 according to the die values 1,2,3,4,5,6.

(b) Both $X$ and $Y$ in the previous example satisfy $(E[X])^2 < E[X^2]$.

9-3:
Choose a random 3-coloring of the elements of $X$ so that each element gets one of the 3 colors independently with probability $1/3$, and let $C$ denote the random variable that counts the number of sets in $F$ that have exactly one element of each color. We have

$$C = \sum_{Y \in F} I_Y,$$

where $I_Y$ denotes the indicator random variable which is 1 if $Y$ has exactly one element of each color and 0 otherwise, for $Y \in F$. We have

$$E[I_Y] = \text{prob}(I_Y = 1) = 3!/3^3,$$

because there are $3^3$ colorings of the elements of $Y$ and $3!$ of them assign each color to exactly one element of $Y$. By the linearity of expectation, we have

$$E(C) = E\left[\sum_{Y \in F} I_Y\right] = \sum_{Y \in F} E[I_Y] = \sum_{Y \in F} \frac{3!}{3^3} = |F| \cdot \frac{3!}{3^3}.$$

It follows that there is a coloring for which $C \geq |F| \cdot 3!/3^3$, that is, at least $|F| \cdot 3!/3^3$ sets in $F$ have exactly one element of each color.

9-4:
Let $X_i$ be 1 if the $i$-th position is a fixed point of the random permutation. Then the expected number of fixed points in a random permutation is $\sum_{i=1}^{n} E[X_i]$.

Since $E[X_i] = \text{prob}(X_i = 1) = 1/n$, we obtain that

$$\sum_{i=1}^{n} E[X_i] = n \cdot 1/n = 1.$$

9-5:
Let $X_{e_1...e_n} = \|\sum_{i=1}^{n} e_i v_i\|$. We choose the weights $e_1, \ldots, e_n$ independently and uniformly at random, and for convenience, we consider the square of the Euclidean norm.

By the linearity of expectation, we obtain that

$$E\left[X_{e_1...e_n}^2\right] = E\left[\left\|\sum_{i=1}^{n} e_i v_i\right\|^2\right] = E\left[\left\langle\sum_{i=1}^{n} e_i v_i, \sum_{i=1}^{n} e_i v_i\right\rangle\right] = E\left[\sum_{i=1}^{n} e_i^2 \|v_i\|^2 + \sum_{i \neq j}^{n} e_i e_j < v_i, v_j >\right] = ...$$
\begin{align*}
&= \mathbb{E} \left[ n + \sum_{i \neq j}^{n} \epsilon_i \epsilon_j < v_i, v_j > \right] = n + \mathbb{E} \left[ \sum_{i \neq j}^{n} \epsilon_i \epsilon_j < v_i, v_j > \right].
\end{align*}

The expected value of the last sum is zero. Indeed, since \( \epsilon_i \) and \( \epsilon_j \) are independent, we have
\[
\mathbb{E} \left[ \sum_{i \neq j}^{n} \epsilon_i \epsilon_j < v_i, v_j > \right] = \sum_{i \neq j}^{n} (\mathbb{E} \epsilon_i)(\mathbb{E} \epsilon_j) < v_i, v_j > = \sum_{i \neq j}^{n} 0 \cdot 0 < v_i, v_j >.
\]

In conclusion, the expected value of the square of the norm is \( n \), so there is at least one choice of the weights for which the vector has norm at least \( \sqrt{n} \).