Cartesian products in combinatorial geometry
Or: Why we are not yet done with Schwartz-Zippel

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What do I mean by *combinatorial geometry*?

Let me introduce it via some central questions (all due to Paul Erdős). All are for finite point sets $P \subset \mathbb{R}^2$.

- How few lines can have exactly 2 points of $P$?
  And how many lines can have at least 3 points of $P$?

- Given $P$ and a set $L$ of lines, how many incidences can there be between $P$ and $L$? In other words, bound
  $$|\{(p, \ell) \in P \times L : p \in \ell\}| = |\{(x, y, a, b) \in P \times L : y = ax + b\}|.$$

- How many times can a distance repeat in $P$?
  I.e., bound $|\{(x, y, s, t) \in P \times P : (x - s)^2 + (y - t)^2 = 1\}|$.

- How many different distances can $P \subset \mathbb{R}^2$ determine?
  I.e., bound $|\{(x - s)^2 + (y - t)^2 : (x, y, s, t) \in P \times P\}|$. 

Lemma (Schwartz-Zippel in $\mathbb{C}^2$)

Let $f \in \mathbb{C}[x, y] \setminus \{0\}$ be a polynomial of degree $d$. Then for any finite sets $A, B \subset \mathbb{C}$ of size $n$ we have

$$|Z(f) \cap (A \times B)| = O_d(n).$$

This bound is tight in the sense that for any $f$ there are $A$ and $B$ of size $n$ for which $|Z(f) \cap (A \times B)| = n$ (pick $n$ points on $Z(f)$ and project).
The Schwartz-Zippel Lemma

**Lemma (Schwartz-Zippel in \( \mathbb{C}^3 \))**

Let \( f \in \mathbb{C}[x, y, z] \setminus \{0\} \) be a polynomial of degree \( d \). Then for any finite sets \( A, B, C \subset \mathbb{C} \) of size \( n \) we have

\[
|Z(f) \cap (A \times B \times C)| = O_d(n^2).
\]

Is this tight?

For \( f = x + y + z \), take \( A = B = \{1, \ldots, n\}, \ C = \{-1, \ldots, -n\} \).

In some (rough) sense, this is the only polynomial for which the bound is tight!
The Elekes-Szabó Theorem

**Theorem (Elekes-Szabó (2004) / Raz-Sharir-De Zeeuw (2014))**

Let \( f \in \mathbb{C}[x, y, z] \) with \( f_x f_y f_z \not\equiv 0 \) and deg \( f = d \). Then for \( A, B, C \subset \mathbb{C} \) of size \( n \) we have

\[
|Z(f) \cap (A \times B \times C)| = O_d(n^{11/6}),
\]

unless around most points of \( Z(f) \) there are an open ball \( D \subset \mathbb{C}^3 \) and three analytic functions \( \varphi_i : \mathbb{C} \to \mathbb{C} \) such that

\[
Z(f) \cap D = \{(x, y, z) \in D : \varphi_1(x) + \varphi_2(y) + \varphi_3(z) = 0\}.
\]
Application of Elekes-Szabó: Expanding polynomials


Let $f \in \mathbb{C}[x, y]$ have degree $d$. Then for any $A, B \subset \mathbb{C}$ of size $n$ we have

$$|f(A \times B)| = \Omega_d(n^{4/3}),$$

unless $f$ has the form

$$g(h(x) + k(y)) \quad \text{or} \quad g(h(x) \cdot k(y)),$$

for $g, h, k \in \mathbb{C}[t]$. 
Application of Elekes-Szabó: Triple lines on curves

- A *triple line* is a line containing at least 3 points of the given point set.
- By double-counting, any $n$ points in $\mathbb{R}^2$ determine at most $\frac{1}{6}n^2$ triple lines.
- Green-Tao (2013): If $n \geq n_K$ points determine more than $\frac{1}{6}n^2 - Kn$ triple lines, then most points lie on a cubic curve.
- Conjecture of Elekes: If $n$ points determine $cn^2$ triple lines, then at least 10 of the points lie on a cubic.

**Theorem (Elekes-Szabó (2013) / Raz-Sharir-De Zeeuw (2014))**

A set of $n$ points on an algebraic curve of degree $d$ in $\mathbb{C}^2$ determines $O_d(n^{11/6})$ triple lines, unless the curve is a cubic.
Schwartz-Zippel bounds $|C \cap (A \times B)|$ for a curve $C$ and “one-dimensional” finite sets $A, B \subset \mathbb{C}$.

What if we consider “two-dimensional” finite sets $P, Q \subset \mathbb{C}^2$? So for a variety $X \subset \mathbb{C}^4$ and $P, Q \subset \mathbb{C}^2$, we’d like to bound

$$|X \cap (P \times Q)|.$$

We can’t always expect a good bound. If $F = xs + yt$ and $P$ is on the $y$-axis, $Q$ on the $s$-axis, then $|Z(F) \cap (P \times Q)| = |P||Q|$. 

Schwartz-Zippel for two-dimensional products
Cartesian polynomials and varieties

**Definition**

Let $G \in \mathbb{C}[x, y] \setminus \mathbb{C}$, $K \in \mathbb{C}[s, t] \setminus \mathbb{C}$. A polynomial $F \in \mathbb{C}[x, y, s, t]$ is $(G, K)$-Cartesian if there are $H, L \in \mathbb{C}[x, y, s, t]$ such that

$$F(x, y, s, t) = G(x, y)H(x, y, s, t) + K(s, t)L(x, y, s, t).$$

**Definition**

A variety $X \subset \mathbb{C}^4$ is Cartesian if there are $G, K$ such that every polynomial defining $X$ is $(G, K)$-Cartesian.

If $X$ is Cartesian, then for $P \subset Z(G), Q \subset Z(K)$ we have

$$|X \cap (P \times Q)| = |P||Q|.$$
Theorem (Nassajian Mojarrad-Pham-Valculescu-Z. (2015))

Let $X \subset \mathbb{C}^4$ be a variety of dimension \textbf{one} or \textbf{two} and degree $d$. Then for all $P, Q \subset \mathbb{C}^2$ of size $n$ we have

$$|X \cap (P \times Q)| = O_d(n),$$

unless $X$ is Cartesian.

Theorem (Nassajian Mojarrad-Pham-Valculescu-Z. (2015))

Let $X \subset \mathbb{C}^4$ be a variety of dimension \textbf{three} and degree $d$. Then for all $P, Q \subset \mathbb{C}^2$ of size $n$ we have

$$|X \cap (P \times Q)| = O_{d,\varepsilon}(n^{4/3+\varepsilon}),$$

unless $X$ is Cartesian.
Schwartz-Zippel for two-dimensional products

- The bound $O_d(n)$ is tight: Take $n$ points on $X$ and project.

- The bound $O_d(n^{4/3})$ is tight for some polynomials. Take
  \[ F = xs - y + t, \quad P = Q = \{(i, j) : 1 \leq i \leq n^{1/3}, 1 \leq j \leq n^{2/3}\}. \]

  For many of the $n^{1/3} \times n^{1/3} \times n^{2/3}$ choices of $x, s, t$, there is a choice of $y$ such that $F = 0$, so $|Z(F) \cap (P \times Q)| = \Omega(n^{4/3})$. We can do the same for any degree.
  
  Valtr (2005) showed $\Omega(n^{4/3})$ for $F = (x - s)^2 + y - t$.

- For $F = xs - y + t$, $O(n^{4/3})$ is of course Szemerédi-Trotter. Indeed, $F(x, y, a, b) = 0$ if and only if $y = ax + b$. 
Corollary: Repeated values of polynomials

The unit distance problem asks to bound $|Z(F) \cap (P \times P)|$ for

$$F = (x - s)^2 + (y - t)^2 - 1.$$ 

Erdős conjectured $O(\varepsilon(|P|^{1+\varepsilon})$. The best bound is still $O(|P|^{4/3})$ due to Spencer, Szemerédi, and Trotter (1984).

With our theorem for three-dimensional varieties, we can generalize this bound to any polynomials for which it is not impossible.

Corollary

Let $F \in \mathbb{C}[x, y, s, t]$ have degree $d$ and $a \in \mathbb{C}$. Then for $P \subset \mathbb{C}^2$

$$|\{(p, p') \in P \times P : F(p, p') = a\}| = O_{d, \varepsilon}(|P|^{4/3+\varepsilon}),$$

unless $F - a$ is Cartesian.
Corollary: Distinct values of polynomials

The distinct distances problem asks to lower bound $|F(P \times P)|$ for $F = (x - s)^2 + (y - t)^2$. Erdős conjectured $\Omega(|P|/\sqrt{\log |P|})$. Moser (1952) proved $\Omega(|P|^{2/3})$, which held until the 80s. Guth and Katz proved $\Omega(|P|/\log |P|)$, and Roche-Newton and Rudnev extended this to Minkowski distances. Otherwise, nothing better than $\Omega(|P|^{2/3})$ is known, even for the dot product $F = xs + yt$.

Our previous corollary gives a weak but very general bound.

**Corollary**

Let $F \in \mathbb{C}[x, y, s, t]$ have degree $d$. Then for $P \subset \mathbb{C}^2$ we have

$$|F(P \times P)| = \Omega_{d,\epsilon}(|P|^{2/3-\epsilon}),$$

unless $F - a$ is Cartesian for some $a \in \mathbb{C}$. 
Repeated and distinct values of polynomial maps

Using our theorem for two-dimensional varieties, we can start to consider the same questions for polynomial maps. For $F_1, F_2 \in \mathbb{C}[x, y, s, t]$, consider $\mathcal{F} : \mathbb{C}^4 \to \mathbb{C}^2$, defined by

$$\mathcal{F}(x, y, s, t) = (F_1(x, y, s, t), F_2(x, y, s, t)).$$

**Corollary**

For $P \subset \mathbb{C}^2$ and $(a, b) \in \mathbb{C}^2$, we have

$$|\{(x, y, s, t) \in P \times P : \mathcal{F}(x, y, s, t) = (a, b)\}| = O_d(|P|),$$

unless $F_1 - a, F_2 - b$ not coprime, or both $(G, K)$-Cartesian.

**Corollary**

For $P \subset \mathbb{C}^2$, we have

$$|\mathcal{F}(P \times P)| = \Omega_d(|P|),$$

unless $F_1, F_2$ are inner equivalent or both $(G, K)$-Cartesian.
A silly proof of one-dimensional Schwartz-Zippel

Let $C$ be a curve of degree $d$ in $\mathbb{C}^2$ and $A, B \subset \mathbb{C}^2$ of size $n$. We can prove $|C \cap (A \times B)| = O_d(n)$ as follows.

For a typical $b \in B$ we have $|C \cap (C \times b)| = O_d(1)$. We can ignore such $b$, and do the same for typical $a \in A$.

We are left with $(a, b)$ for which $C$ contains $C \times b$ and $a \times C$. If there are at most $d$ such $(a, b)$ we are done. Otherwise, we have $A' \times B' \subset C$ with $|A'|, |B'| > d$.

**Lemma (Alon’s Combinatorial Nullstellensatz in $\mathbb{C}^2$)**

Let $f \in \mathbb{C}[x, y]$, and $g \in \mathbb{C}[x]$, $k \in \mathbb{C}[y]$ squarefree. Then $Z(g) \times Z(k) \subset Z(f)$ if and only if $f = g(x)h(x, y) + k(y)l(x, y)$, with $\deg(g) + \deg(h) \leq \deg(f)$ and $\deg(k) + \deg(l) \leq \deg(f)$.

Choose $Z(g) = A'$ and $Z(k) = B'$, with $\deg(g), \deg(k) > d$. Contradiction.
Sketch of the proof for two-dimensional $X \subset \mathbb{C}^4$

For $q \in Q \subset \mathbb{C}^2$, the fiber $X \cap (\mathbb{C}^2 \times q)$ is typically a finite set of size $d$, so all typical fibers give $O_d(|Q|)$ pairs $(p, q) \in X \cap (P \times Q)$.

The $q \in \mathbb{C}^2$ for which the fiber is not typical lie in a curve $W \subset \mathbb{C}^2$. Similarly, untypical $p$ lie in a curve $V$. Assume $V, W$ irreducible. We can reduce to the case $P' \times Q' \subset V \times W$.

For $F \in I(X)$ and $q \in W$, set $C_q = \{p : F(p, q) = 0\}$. The $q \in W$ for which $C_q \cap V$ is finite give altogether $O_d(|Q|)$. For other $q$ we have $V \subset C_q$. Same for $p \in V$.

We are done unless there are large $I \subset P$, $J \subset Q$ such that

$$I \times J \subset X.$$
Nullstellensätze for two-dimensional products

**Lemma (Alon’s Nullstellensatz (sort of) in \( \mathbb{C}^2 \))**

Let \( f \in \mathbb{C}[x, y] \) and \( g \in \mathbb{C}[x], \ k \in \mathbb{C}[y] \) squarefree. Then \( Z(g) \times Z(k) \subset Z(f) \) if and only if \( f = g(x)h(x, y) + k(y)l(x, y) \).

**Lemma (NPVZ15)**

Let \( F \in \mathbb{C}[x, y, s, t] \) and \( G \in \mathbb{C}[x, y] \setminus \mathbb{C}, \ K \in \mathbb{C}[s, t] \setminus \mathbb{C} \) squarefree. Then \( Z(G) \times Z(K) \subset Z(F) \) if and only if we can write \( F = G(x, y)H(x, y, s, t) + K(s, t)L(x, y, s, t) \), i.e., \( F \) is Cartesian.

**Lemma (NPVZ15)**

Let \( F \in \mathbb{C}[x, y, s, t] \) have degree \( d \). Then \( F \) is Cartesian if and only if there are \( I, J \subset \mathbb{C}^2 \) of size \( > d^2 \) such that \( I \times J \subset Z(F) \).

With the third lemma the proof for two-dimensional \( X \) is done.
Proof sketch of our two-dimensional Nullstellensatz

Given $F$, for $q \in \mathbb{C}^2$ we define

$$C_q = \{ p : F(p, q) = 0 \},$$

and for $p \in \mathbb{C}^2$ we define

$$C_p^* = \{ q : F(p, q) = 0 \}.$$

_Duality_: $p \in C_q$ if and only if $q \in C_p^*$.

**Lemma**

Let $F \in \mathbb{C}[x, y, s, t]$ have degree $d$. Then $F$ is Cartesian if and only if there are $I, J \subset \mathbb{C}^2$ of size $> d^2$ such that $I \times J \subset \mathbb{Z}(F)$.

**Proof.** Write $C_q = \{ p : F(p, q) = 0 \}$ and $C_p^* = \{ q : F(p, q) = 0 \}$. $I \subset C_q \ \forall q \in J \Rightarrow \exists C \subset \cap_{q \in J} C_q \Rightarrow F(p, q) = 0 \ \forall p \in C, q \in J$

$\Rightarrow J \subset C_p^* \ \forall p \in C \Rightarrow \exists C^* \subset \cap_{p \in C} C_p^*$

$\Rightarrow F(p, q) = 0 \ \forall p \in C, q \in C^* \Rightarrow C \times C^* \subset \mathbb{Z}(F)$

If $C = \mathbb{Z}(G)$ and $C^* = \mathbb{Z}(K)$, then we get $F = GH + KL$. □
Sketch of the proof for three-dimensional $X = Z(F) \subset \mathbb{C}^4$

Here a typical fiber $Z(F) \cap (\mathbb{C}^2 \times q)$ is a curve. We want to bound the incidences between $P$ and the curves

$$C_q = \{ p \in \mathbb{C}^2 : F(p, q) = 0 \} \text{ for } q \in Q.$$ 

**Theorem (ST→PS→SZ→NPVZ; see also WYZ, FPSSZ)**

Let $F \in \mathbb{C}[x, y, s, t]$ have degree $d$. Let $P, Q \subset \mathbb{C}^2$ have size $n$ and define $C_Q = \{ C_q : q \in Q \}$. If the incidence graph of $P$ and $C_Q$ contains no $K_{M,M}$, then the number of incidences satisfies

$$I(P, C_Q) = O_{d,M,\varepsilon}(n^{4/3+\varepsilon}).$$

We have $I(P, C_Q) = |Z(F) \cap (P \times Q)|$, so we are done unless the incidence graph contains a $K_{M,M}$ with $M > d^2$. But this means $I \times J \subset Z(F)$ with $|I|, |J| > d^2$, which by our Nullstellensatz for two-dimensional products implies that $F$ is Cartesian.
Papers discussed
