Graph theory - solutions to problem set 6

Exercises

1. Determine the chromatic number of the first graph and the edge-chromatic number of the second graph below.

Solution:

The chromatic number of the left graph and the edge-chromatic number of the right graph are both 4. Shown are 4-colorings for both.

To show that the coloring of the first graph is optimal, we try to 3-color it. Start with the outer $C_5$: up to isomorphism there is only one coloring, red-blue-red-blue-green. This forces a red, blue, and green vertex on the inner ring of 5 vertices, which forces a fourth color on the middle vertex.

By Vizing’s theorem we cannot color edges of the right graph by less than 4 colors.

2. For a graph $G$, we define $G[X]$, the subgraph induced by the vertex set $X \subseteq V(G)$ as the graph with vertex set $X$ that contains all the edges of $G$ with both ends in $X$.

Prove that $\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$.

Solution: Define $\chi_1 = \chi(G[X])$, $\chi_2 = \chi(G[V \setminus X])$. We prove that there is a valid coloring of $G$ with $\chi_1 + \chi_2$ colors: Color the vertices of $X$ with $\chi_1$ colors such that we have a valid coloring of $G[X]$, and color $V \setminus X$ with $\chi_2$ colors different from the first $\chi_1$ colors so that we get a valid coloring of $G[V \setminus X]$. Note that the edge $e$ of $G$ is either fully included in one of $G[X]$ or $G[V \setminus X]$, or it connects them. In both cases the end vertices of $e$ get different colors: since in the former case both of the induced subgraphs have a valid coloring, and in the latter one, it follows from the fact that the vertices in $X$ get different colors from the vertices in $V \setminus X$.

3. Are the following statements true?

(a) If $G$ and $H$ are graphs on the same vertex set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$.
(b) Every graph $G$ has a coloring with $\chi(G)$ colors where $\alpha(G)$ vertices get the same color.

Solution: We give counterexamples to both of the statements:

(a) Let $n \geq 6$ be a positive integer and $V$ be a set of $n$ vertices with the partition $V = V_1 \cup V_2$ such that $|V_1| = \lfloor \frac{n}{2} \rfloor$ and $|V_1| = \lceil \frac{n}{2} \rceil$. Let $G, H$ be the graphs with the vertex set $V$ satisfying $E(G) = \{vw : v, w \in V_1 \text{ or } v, w \in V_2\}$, $E(H) = \{vw : v \in V_1, w \in V_2\}$.

Note that $G$ is the union of the complete graphs on $V_1$ and $V_2$, so we have $\chi(G) \leq \max\{|V_1|, |V_2|\} = \lceil \frac{n}{2} \rceil$. On the other hand, $H$ is a bipartite graph so $\chi(H) = 2$. Therefore, we have $\chi(G \cup H) = \chi(K_n) = n > \lceil \frac{n}{2} \rceil + 2 = \chi(G) + \chi(H)$.
(b) Let $n > 1$ be a positive integer. Consider the double star graph $G$, i.e. $G$ is the union of two disjoint star graphs $K_{1,n}$, such that the two centers of the stars are connected:

$$V = \{v_0, v_1, \ldots, v_n, u_0, u_1, \ldots, u_n\}, \quad E = \{v_0v_i : 1 \leq i \leq n\} \cup \{u_0u_i : 1 \leq i \leq n\} \cup \{v_0u_0\}.$$ 

It is easy to check that $\alpha(G) = |V(G)| - 2$, since the set of vertices $\{v_1, \ldots, v_n, u_1, \ldots, u_n\}$ is independent, and any independent set of vertices may have at most one of $u_0$ or $u_i$ for any $i$, and at most one of $v_0$ or $v_i$ for any $i$. Furthermore, the set $\{v_1, \ldots, v_n, u_1, \ldots, u_n\}$ is the unique maximal independent set of vertices in $G$. On the other hand, we have $\chi(G) = 2$ since we can partition $V(G)$ into independent subsets $\{u_0, v_1, \ldots, v_n\}, \{v_0, u_1, \ldots, u_n\}$. Note that if we color all the vertices in $\{v_1, \ldots, v_n, u_1, \ldots, u_n\}$ with the same color, we need 3 colors to have a valid coloring of $G$. So this gives a counterexample to the statement.

**Problems**

4. (a) Find the edge-chromatic number of $K_{2n+1}$ (don’t use Vizing’s theorem).
   (b) Find the edge-chromatic number of $K_{2n}$.

**Solution:** $\chi'(K_{2k-1}) = 2k - 1$: To get a $2k - 1$-coloring, place the vertices $v_i$ on a circle with equal spacing. Then for each vertex $v_i$, give the same color to the edges $v_{i-1}v_i, v_{i-2}v_{i+2}$, etc. (these edges will be parallel). This way we color all the edges with $2k - 1$ colors.

Suppose we could color the edges of $K_{2k-1}$ with $2k - 2$ colors. Each color class has at most $k - 1$ edges, so with $2k - 2$ colors we can color at most $(2k-2)(k-1)$ edges. But $K_{2k-1}$ has $\binom{2k-2}{2} = (2k-1)(k-1)$ edges, so this can’t work.

$\chi'(K_{2k}) = 2k - 1$: Now place $2k - 1$ of the vertices $v_i$ on a circle with equal spacing, and put the remaining vertex $u$ at the center of the circle. Then for each $v_i$, color in the same way as in the odd case, and also give that color to the edge $uv_i$. This gives an edge coloring with $2k - 1$ colors.

In this case we have $\Delta(K_{2k}) = 2k - 1$, so by a theorem from class there is no edge coloring with fewer colors.

5. Let $G$ be a graph on $n$ vertices and $\overline{G}$ be its complement. Prove that
   (a) $\chi(G)\chi(\overline{G}) \geq n$.
   (b) $\chi(G) + \chi(\overline{G}) \leq n + 1$.

**Solution:**

(a) Note that the union $G \cup \overline{G}$ is the complete graph $K_n$. We construct a valid coloring of $K_n$ with $\chi(G)\chi(\overline{G})$ colors. Then we get the required inequality, since $\chi(K_n) = n$. Denote $V = V(G)$. Let $c : V \rightarrow \{1, 2, \ldots, \chi(G)\}$ be a valid coloring of $G$ and $\overline{c} : V \rightarrow \{1, 2, \ldots, \chi(\overline{G})\}$ be a valid coloring of $\overline{G}$. Define the coloring $c' : V \rightarrow \{1, 2, \ldots, \chi(G)\} \times \{1, 2, \ldots, \chi(\overline{G})\}$ with $c'(v) = (c(v), \overline{c}(v))$ for $v \in V$. It is easy to see that $c'$ is a valid coloring for $K_n$ on $V$: for distinct vertices $u, v \in V$, if $uv \in E(G)$, then $c(u) \neq c(v)$, and if $uv \in E(\overline{G})$, then $\overline{c}(u) \neq \overline{c}(v)$, both of which imply that $c'(u) \neq c'(v)$.

(b) We prove it by induction on the number of the vertices. It is easy to check the induction basis. Now suppose the inequality holds for all graphs with $n$ vertices, we prove it for the graph $G$ on $n + 1$ vertices. Fix the vertex $v \in V(G)$ and let $k$ be its degree in $G$, so the degree of $v$ in $\overline{G}$ is $n - k$. Consider the graph $G - v$. Note that adding back $v$ to $G - v$ does not increase the chromatic number if $\chi(G - v) > k$, since one can color $v$ by an existing color different from the colors of its $k$ neighbors; otherwise, it will increase the chromatic number by at most one. The same statement holds for $\overline{G} - v$ with the condition $\chi(\overline{G} - v) > n - k$. Therefore, if at least one of $\chi(G - v) > k$ and $\chi(\overline{G} - v) > n - k$ holds, then applying induction hypothesis to $G - v$ will complete the proof:

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G} - v) + 1 \leq n + 2.$$
Otherwise, we have \( \chi(G - v) \leq k \) and \( \chi(G) \leq n - k \), which implies
\[
\chi(G) + \chi(G) \leq \chi(G - v) + \chi(G) + 2 \leq k + n - k + 2 = n + 2.
\]

This finishes the proof.

6. (a) Show that if an \( n \)-vertex graph is \( d \)-degenerate, then it has at most \( dn \) edges.
(b) Prove that if the longest path in \( G \) has length \( \ell \), then \( \chi(G) \leq \ell + 1 \).

**Solution:**

(a) This can be proved by induction on the number of vertices \( n \). Note that by the induction hypothesis, removing a vertex of degree at most \( d \) would result in a graph with at most \( d(n - 1) \) edges, so the original graph has at most \( d(n - 1) + d = dn \) edges.

(b) This follows from the fact that any such graph is \( \ell \)-degenerate. To see this, let \( G' \) be a subgraph of \( G \) and \( v \) be an endpoint of a longest path in \( G' \). Since this path cannot be extended, all the neighbors of \( v \) in \( G' \) are contained in this path, therefore \( \deg_{G'}(v) \leq \ell \).

7. Let \( G \) be a 3-regular graph with \( \chi'(G) = 4 \). Prove that \( G \) does not have a Hamilton cycle.

**Solution:** Note that since \( \sum d(v) = 2|E(G)| \) and every \( d(v) = 3 \), \( G \) must have an even number of vertices. Suppose that \( G \) has a Hamilton cycle \( C \). It must be even, so we can color it with 2 colors. Every vertex has 2 edges from \( C \) and one other edge, so the edges not in \( C \) form a matching. Hence we can color these edges with one color. This gives an edge-coloring of \( G \) with 3 colors, contradicting \( \chi'(G) = 4 \).

8. Prove that if every two odd cycles of \( G \) intersect in at least one vertex, then \( \chi(G) \leq 5 \).

**Solution:** If \( G \) has no odd cycles, then \( G \) is bipartite, which means that \( \chi(G) \leq 2 \). Thus we can assume that \( G \) has at least one odd cycle.

Let \( C \) be any odd cycle, and remove its vertices from \( G \) to get a new graph \( G - C \). It has no odd cycles, since every odd cycle previously intersected \( C \). This implies that \( G - C \) is bipartite, or in other words 2-colorable. Then we can combine a 2-coloring of \( G - C \) with a 3-coloring of \( C \) to get a 5-coloring of \( G \).