Graph theory - solutions to problem set 2

Exercises

1. Prove the triangle-inequality in graphs: for any three vertices \( u, v, w \) in a graph \( G \),
\[
d(u, v) + d(v, w) \geq d(u, w).
\]

**Solution.** If \( d(u, v) = \infty \) or \( d(v, w) = \infty \), there is nothing to prove.
Otherwise, according to the definition of the distance, there is a \( u-v \) path of the length \( d(u, v) \) and a \( v-w \) path of the length \( d(v, w) \). Joining them together we obtain the \( u-w \) walk of the length \( d(u, v) + d(v, w) \).
We have seen in class, that this walk will then contain a \( u-w \) walk, which is clearly not longer than the walk. Therefore, the shortest \( u-w \) path is no longer than \( d(u, v) + d(v, w) \).

2. Show that a graph is connected if and only if it contains a spanning tree.

**Solution.** If there is a spanning tree then the graph is clearly connected: for any vertices \( u \) and \( v \), there will be a \( u-v \) path in the tree, hence in the graph, as well. If the graph is connected then the BFS algorithm finds a spanning tree, and this proves that a spanning tree exists.

3. Prove that a forest on \( n \) vertices with \( c \) connected components has exactly \( n - c \) edges.

**Solution.** Let \( T_1, \ldots, T_c \) be the components of the forest, on \( n_1, \ldots, n_c \) vertices, respectively. Each \( T_i \) itself is a connected acyclic graph, hence it is a tree (considered as a graph on its own). Therefore, \( T_i \) contains \( n_i - 1 \) edges for each \( i \). Altogether, the graph contains \( \sum_{i=1}^{c} (n_i - 1) = \sum_{i=1}^{c} n_i - c = n - c \) edges.

4. Let \( T \) be a tree and \( e \) be an edge of \( T \). Prove that \( T - e \) is not connected.

**Solution.** Let \( e = uv \) and suppose \( T - e \) is connected. Then, in particular, \( T - e \) contains a \( u-v \) path \( P \). But then \( P + e \) is a cycle in \( T \), a contradiction.

5. Let \( T \) be a tree and let \( u \) and \( v \) be two non-adjacent vertices of \( T \). Prove that \( T + uv \) contains a unique cycle.

**Solution.** \( T \) contains a \( u-v \) path and adding \( uv \) to it will form a cycle, so \( T + uv \) contains at least one cycle. Suppose it has two different cycles. Both of them must contain \( uv \), otherwise, removing \( uv \), we would get a cycle in \( T \). But then if we remove \( uv \), we get two different \( u-v \) paths in \( T \), which contradicts a result from the lecture. Hence there is a unique cycle.

Problems

6. Let \( G \) be a graph on \( n \) vertices. Prove that

(a) if \( G \) has \( dn \) edges, then it contains a path of length at least \( d \).
(b) if \( G \) has at least \( 2n - 1 \) edges, then it contains an even cycle.

**Solution.**

(a) On the lecture we proved that \( G \) contains a subgraph \( H \) with \( \delta(H) > d \). By another result from the lecture, \( H \) contains a path of length at least \( \delta(H) \).
(b) On the lecture we proved that \( G \) contains a bipartite subgraph \( H \) with \( |E(H)| \geq \frac{|E(G)|}{2} \). We have that \( |V(H)| \leq n \) and \( |E(H)| \geq n \) in \( H \). Therefore, \( H \) contains a cycle (otherwise \( H \) is a forest, and the number of edges in a forest is strictly less than the number of vertices). Since \( H \) is a bipartite graph, this cycle has even length.
7. Let $W$ be a closed walk that uses the edge $e$ exactly once. Prove that $W$ contains a cycle through $e$.

**Solution.** Let $v_1 v_2 \ldots v_n v_1$ be a shortest closed walk that uses the edge $e$ exactly once. We claim that this walk is a cycle. Indeed, if $v_i = v_j$ for some $i < j$, then either the closed walk $v_1 \ldots v_i v_{j+1} \ldots v_1$ or the closed walk $v_i v_{i+1} \ldots v_j$ uses the edge $e$ exactly once, and both of them are shorter, which is not possible. (Why doesn’t this argument work for an arbitrary walk that uses the edge $e$ exactly twice?)

8. Prove that every connected graph on $n \geq 2$ vertices has a vertex that can be removed without disconnecting the remaining graph.

**Solution.** Take a spanning tree $T$ of the graph. It has at least two leaves, say $x$ and $y$. Then $T - x$ and $T - y$ are both connected, hence so are their supergraphs, $G - x$ and $G - y$.

9. Let $T$ be a tree on $n$ vertices that has no vertex of degree 2. Show that $T$ has more than $n/2$ leaves.

**Solution.** $T$ has $n - 1$ edges, so the sum of the degrees is $2n - 2$. Suppose $T$ has at most $n/2$ leaves. Then at least $n/2$ vertices have degree at least 3. But then the sum of the degrees is at least $1 \cdot \frac{n}{2} + 3 \cdot \frac{n}{2} = 2n$, which is a contradiction.

10. Show that every tree $T$ has at least $\Delta(T)$ leaves.

**Solution.** Let $v$ be a vertex with degree $d = \Delta(T)$. For every edge $vw$ incident to $v$, take a longest path starting with $vw$. By maximality (as in the proof that every tree has a leaf), the last vertex of this path is a leaf. Doing this for each of the $d$ edges incident to $v$, we get $d$ paths starting at $v$, which are disjoint except for $v$ (otherwise we would get a cycle). Thus each path gives a different leaf, and we get $d = \Delta(T)$ leaves.

**Alternative solution:** If you remove $v$ and its incident edges, you are left with $d$ connected components $T_1, \ldots, T_d$, each of which is a tree. By a lemma from class, every tree with at least two vertices has at least two leaves. Hence the $T_i$ with at least two vertices have at least two leaves, one of which must be a leaf of $T$ (one of the two leaves might have been adjacent to $v$, but not both because that would give a cycle). Some of the $T_i$ might be single vertices, in which case those vertices were leaves in $T$ (they must have been adjacent to $v$ and to no other vertex).

11. Let $T$ be an $n$-vertex tree that has exactly $2k$ vertices of odd degree. Show that $T$ can be split into $k$ edge-disjoint paths (i.e., $T$ is the union of $k$ edge-disjoint paths).

**Solution.** We prove a more general statement: the above claim is true for forests, not only trees. Making our problem more general allows us to use a simpler induction argument.

So let us do induction on $k$. For $k = 0$ our forest is empty (every nonempty forest has a leaf, thus an odd-degree vertex), so the statement holds. Now assume we know it for $k$, and take a forest $T$ with $2k + 2$ odd-degree vertices. Let $P$ be a maximal path in $T$. We have seen in class that $P$ will connect two leaves. We claim that if we delete the edges of $P$ then we get a forest $T - P$ with $2k$ odd-degree vertices. Indeed, the two leaves will lose the edge touching them, so they have degree 0 in $T - P$, while every other vertex loses either 0 or 2 incident edges, hence the parity of its degree does not change. In other words, we lost two odd-degree vertices and did not gain anything. So we can apply induction on $T - P$ to get $k$ paths partitioning its edge set. Together with $P$ we have $k + 1$ paths partitioning the edge set of $T$, which is what we wanted to show.