Solutions 3.

3-1: We seek injective functions \( f : \{1, \ldots, n\} \to \{1, \ldots, m\} \). For the first element, we have \( m \) choices to associate a value; for the second element we are left with \( n - 1 \) possibilities and so on. For the \( n \)th element we are left with \( m - n + 1 \) possibilities. That means that the total number of injective functions is

\[
\frac{m!}{(m - n)!}.
\]

3-2: The problem is equivalent to finding the permutations with no fixed point (we consider a fixed point if a man dances with his wife). We count the number of permutations with at least a fixed point, using the inclusion and exclusion principle and we subtract this value from the total number of permutations.

First, note that the total number of pairings is \( n! \).

Let \( A_i \) be the set of pairings for which the \( i \)th man dances with his wife.

In general, let \( A_{i_1, \ldots, i_k} \) be the set of pairings for which the men \( i_1, \ldots, i_k \) dance with their respective wives.

Note that \( |A_{i_1, \ldots, i_k}| = (n - k)! \).

By inclusion and exclusion, the number of pairings for which at least one man dances with his wife is

\[
\sum_{i=1}^{n} |A_i| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1, i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1, i_2, i_3}| - \ldots + (-1)^{n+1} |A_{1,2,\ldots,n}| =
\]

\[
= n(n - 1)! - \frac{n}{2} (n - 2)! + \frac{n}{3} (n - 3)! - \ldots + (-1)^{n+1} \binom{n}{n} =
\]

\[
= n! - \frac{1}{2!} n! + \frac{1}{3!} n! - \ldots + (-1)^{n+1} \frac{1}{n!} n!.
\]

Recall that we were interested in the number of pairs for which no man dances with his wife, so, as said in the remark above we have to subtract the above value from the total number of pairings. That means, the desired result is

\[
\sum_{i=1}^{n} |A_i| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1, i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1, i_2, i_3}| - \ldots + (-1)^{n+1} |A_{1,2,\ldots,n}| =
\]

\[
= n! - \frac{1}{2!} n! + \frac{1}{3!} n! - \ldots + (-1)^{n+1} \frac{1}{n!} n!.
\]

As said before, this result is the number of permutations of a set of \( n \) elements with no fixed point.

3-3:

a) Let \( A_i \) be the set of permutations which fix only \( i \). Since \( A_i \) has no other fixed points apart from \( i \), we obtain that its cardinality is equal to the cardinality of the set of all the permutations of an \( n - 1 \)-element set with no fixed point (in fact, the set of the other elements except \( i \)). But, from exercise 2, we already know that this number for \( n - 1 \) is
\[(n-1)! \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!}\right)\].

On the other hand, there are \(n\) choices for the fixed point and no two such permutations can coincide (since it is only one fixed point). That means, the desired result is

\[n! \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!}\right).\]

b) Applying the same idea as for a), we obtain

\[\binom{n}{k} (n-k)! \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-k} \frac{1}{(n-k)!}\right).\]

3-4: First, let us count the number of dominoes containing numbers from 0 to 8 and admitting doubles. There are two cases with respect to the values that we can write on a domino piece: either the two values are equal (doubles), or they are different. The number of doubles is 9.

Now note that the number of non-doubles is \(9 \cdot \frac{8}{2} = 36\), because we need to choose two numbers out of 9. On the other side, performing this trick we count twice some domino pieces (for instance 1, 2 we count it also as 2, 1).

That gives us in total 45 possible pieces in the first domino set.

For the second, since we count only those pieces that do not contain doubles, and the numbers on each domino piece can be chosen from the set \(\{0, \ldots, 9\}\), by an argument analogous to the one before, we obtain only 45 possible pieces. Thus the two sets are of the same size.

3-5: Note that \(10^{40} = 2^{40} \cdot 5^{40}\) and \(20^{30} = 2^{60} \cdot 5^{30}\). Consider the following sets

\[A = \{a \in \mathbb{N} : a|10^{40}\}, \quad B = \{a \in \mathbb{N} : a|20^{30}\}.\]

We have to count \(|A \cup B|\). By inclusion-exclusion principle, we have

\[|A \cup B| = |A| + |B| - |A \cap B|.

Remember that the number of divisors of \(n = p_1^{a_1} \ldots p_k^{a_k}\) is \((a_1 + 1) \cdot \ldots \cdot (a_k + 1)\). That means, \(|A| = 41 \cdot 41 = 1681\) and \(|B| = 61 \cdot 31 = 1891\).

On the other hand,

\[|A \cap B| = \{a \in \mathbb{N} : a|10^{40} \text{ and } a|20^{30}\} = \{a \in \mathbb{N} : a|2^{40} \cdot 5^{30}\}.\]

That means, \(|A \cap B| = 41 \cdot 31 = 1271\).

Altogether, we obtain
\[ |A \cup B| = 1681 + 1891 - 1271 = 2301. \]

**3-6:** \( 385 = 5 \times 7 \times 11. \) Using Euler’s function (recall its definition), we obtain that the result is
\[ \phi(385) = 385 \cdot \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \left(1 - \frac{1}{11}\right) = 4 \cdot 6 \cdot 10 = 240. \]

**3-7:** *Idea 1.* Let \( v \) be the vertex of degree \( n \), and \( v_1, \ldots, v_n \) be its neighbors. Then from \( v \), we can find \( n \) paths starting at \( v \) in the direction of \( v_1, \ldots, v_n \). Since a tree has no cycles, no two such paths intersect and none of them intersects itself. Thus, each of these paths ends in a vertex of degree 1, so since there are \( n \) such paths, there are at least \( n \) vertices of degree 1 in the tree.

*Idea 2.* Let \( T = (V, E) \) be the tree. Consider the following sets:
\[ V_1 = \{ v \in V : d(v) = 1 \}, \quad V_2 = \{ v \in V : d(v) \geq 2 \}, \]
where \( d(v) \) denotes the degree of vertex \( v \).

Since \( T \) is a tree (and thus connected), every vertex of \( T \) is either in \( V_1 \) or in \( V_2 \), but not in both (the degree is unique, and since it is connected the degree is at least 1).

Recall that, since \( T \) is a tree, the following holds
\[ \sum_{v \in V} d(v) = 2(|V| - 1). \]

On the other hand, using the fact that \( V = V_1 \cup V_2 \) and \( V_1 \cap V_2 = \emptyset \), we have
\[ \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = \sum_{v \in V} d(v) = 2(|V| - 1). \]

Let \( u \) be the vertex of \( T \) of degree \( n \). Then obviously, \( u \in V_2 \). Also, since the degree of every vertex in \( V_1 \) is 1, we have
\[ \sum_{v \in V_1} d(v) = 1 \cdot |V_1| = |V_1|. \]

On the other hand, since the degree of each vertex in \( V_2 \) is at least 2, we have
\[ \sum_{v \in V_2} d(v) = d(u) + \sum_{v \in V_2 \setminus \{u\}} d(v) \geq d(u) + 2 \cdot |V_2 \setminus \{u\}| = n + 2 \cdot (|V| - 1 - |V_1|). \]

Thus
\[ |V_1| + n + 2(|V| - 1 - |V_1|) \leq 2(|V| - 1), \]
and thus
\[ |V_1| \geq n. \]

That means, there are at least \( n \) vertices of degree 1, which completes the proof.
3-8: We partition the set \([n]\) into disjoint subsets as follows.

\[ S_d := \{1 \leq x \leq n: \gcd(x, n) = d\}, \text{ for all } d|n. \]

We now prove that

\[ |S_d| = \phi\left( \frac{n}{d} \right). \]

For any \(d|n\), define

\[ S'_d := \left\{1 \leq x \leq n/d: \gcd\left(\frac{x}{d}, \frac{n}{d}\right) = 1\right\}. \]

Note that \(x \in S_d\) if and only if \(x/d \in S'_d\). Thus,

\[ |S_d| = |S'_d| = \phi(n/d), \]

which implies that

\[ n = \sum_{d|n} |S_d| = \sum_{d|n} |S'_d| = \sum_{d|n} \phi(n/d) = \sum_{d|n} \phi(d). \]

This completes the proof.

3-9: We may assume that the numbers form an increasing sequence: \(1 \leq a_1 < a_2 < \ldots < a_{20} \leq 70\). Clearly, \(a_1 + (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \ldots (a_{20} - a_{19}) = a_{20} \leq 70\). On the other hand, the 19 parentheses above are 19 differences. Suppose for a contradiction that no difference appears more than 3 times. Then the left hand side above is at least:

\[ 1 + 3 \times 1 + 3 \times 2 + 3 \times 3 + 3 \times 4 + 3 \times 5 + 3 \times 6 + 1 \times 7 = 71, \]

which is greater than 70, a contradiction.