Solutions 2.

2-1:
(a) By the multinomial theorem, we get $\frac{9!}{3!2!}$ anagrams. Another way of seeing this: If the 9 letters were distinct, we would have 9!. We can order the three $x$s in 3! ways, and the two $y$s in 2 ways. So we counted each anagram $3! \cdot 2$ times.
(b) We consider the three copies of "xxx" as one letter and use the multinomial theorem to get $\frac{7!}{2!}$ anagrams.
(c) First, we ignore the other letters and determine the number of patterns consisting of three copies of 'x' and two copies of 'y' provided that each copy of 'y' occurs between two copies of 'x'. Note that we have three such patterns: "xyxyx", "xyxx", "xyyxx". Now we find the number of words out of the letters "a,b,c,d,x,x,x,y,y" where the letters 'x' and 'y' form one of the three described patterns. Each of these patterns occurs in $\frac{9!}{5!}$ words, so in total we have $3 \cdot \frac{9!}{5!}$ words.

2-2:
(a) We have
$$\frac{(4n)!}{2^{3n} \times 3^n} = \frac{(4n)!}{(4!)^n}.$$ Since $\frac{(4n)!}{(4!)^n}$ is a multinomial coefficient, it is an integer. Therefore, $(4n)!$ is divisible by $2^{3n} \times 3^n$.
(b) Let the consecutive numbers be $n,n-1,\ldots,n-r+1$. We have
$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \binom{n}{r},$$ which implies the statement.

2-3:
(a) $\binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k}$. If $k < \lfloor n/2 \rfloor$ then $n-k \geq \lfloor n/2 \rfloor \geq k+1$ and therefore $\frac{n-k}{k+1} \geq 1$.
(b) Note that $\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} = 2^n$. This immediately gives $\binom{n}{\lfloor n/2 \rfloor} \leq 2^n$, but we need to show the upper bound $2^{n-1}$. Recall that
$$\sum_{i \text{ even}} \binom{n}{i} = \sum_{i \text{ odd}} \binom{n}{i} = 2^{n-1}.$$ As $\binom{n}{\lfloor n/2 \rfloor}$ is a term in one of these sums (depending on the parity of $\lfloor n/2 \rfloor$), it must be at most $2^{n-1}$.
To show the lower bound, note that part (a) implies that $\binom{n}{\lfloor n/2 \rfloor}$ is a largest binomial coefficient of the form $\binom{n}{i}$. Indeed, it is at least $\binom{n}{i}$ for $i \leq n/2$ by (a). On the other hand, the binomial coefficients are symmetric: $\binom{n}{i} = \binom{n}{n-i}$, so we have the same numbers for $i > n/2$, all at most $\binom{n}{\lfloor n/2 \rfloor}$. This shows that $\binom{n}{\lfloor n/2 \rfloor}$ must be at
least the average of the \( \binom{n}{i} \), which is

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\frac{\sum_{i=0}^{n} \binom{n}{i}}{n+1} = \frac{2^n}{n+1}.
\]

**2-4:** We will use that if \( f(x) \) and \( g(x) \) are differentiable functions such that \( f(0) = g(0) \) and \( f'(x) \geq g'(x) \) for all \( x \geq 0 \), then \( f(x) \geq g(x) \geq 0 \) for all \( x \geq 0 \). Here \( e^0 = 0 + 1 = 1 \) and \( (e^x)' = e^x \), a monotone increasing function, so \( e^x \geq e^0 = 1 = (x+1)' \) for any \( x \geq 0 \).

**2-5:**
(a) \( n^{\alpha}/n^{\beta} = n^{\alpha-\beta} \). If \( \alpha \leq \beta \) then the exponent is non-positive, hence \( n^{\alpha-\beta} \leq 1 \). This implies \( n^\alpha = O(n^\beta) \).
(b) We will show that \( 5^n \geq n^5 \) if \( n \) is sufficiently large.

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5^n \geq n^5 \iff n \ln 5 \geq 5 \ln n.
\]

Let \( f(x) = x \ln 5 - 5 \ln x \). Then we have \( f'(x) = \ln 5 - \frac{5}{x} \geq 0 \) if \( x \geq 5 \) and thus \( f(x) \) is increasing for \( x \geq 5 \) where we also used that \( f(5) = 0 \). This yields \( f(x) \geq 0 \) if \( x \geq 5 \).
(c) This part can be proved in the same way as part (b).

**2-6:** By symmetry, it is enough to prove \( f(n) = O(g(n)) \). We prove \( f(n) \leq \frac{2n}{x} g(n) \) for large enough \( n \). Indeed, \( \frac{2n}{x} g(n) - f(n) = an^2 + \hat{b}n + \hat{c} \) where \( \hat{b}, \hat{c} \) are some real numbers. As \( a \) is positive, this quadratic function is positive for large enough \( n \).

**2-7:** Answer: \( \frac{(n-1)!}{2} \). We will number the seats from 1 to \( n \). Since rotating a seating order yields an equivalent seating order, we may assume that person number 1 sits at seat number 1. The other people can be seated in \( (n-1)! \) ways. On the other hand, if we reverse an order, we obtain an equivalent order, since for each person, the set of two neighbors remains the same. So, we counted each seating order twice.