Graph Theory: Problem set 7

April 12, 2010

1. An $n$ by $n$ matrix with entries from \{1, 2, \ldots, n\} is called a Latin square, if every element of \{1, 2, \ldots, n\} appears exactly once in each column, and exactly once in each row. Recast the problem of constructing Latin squares as a coloring problem.

2. By an outerplanar graph we understand a planar graph having all the vertices on the outer face.

Show that the chromatic number of an outerplanar graph is at most three.

3. Let $G$ denote a graph that contains a cycle. Show that $\chi(G) = \max_{C \subseteq C(G)} \chi(C)$, where $C(G)$ is the set consisting of the vertex two-connected components (subgraphs) in $G$.

4. Let $k$ denote a natural number. Describe a construction of a triangle-free graph with chromatic number $k$ (a hint is present on this page).

5. We define the line graph $G' = (E', E')$ of $G$ to be the graph whose vertex set is simply the edge set of $G$ and two vertices in $G'$ are joined by an edge if their corresponding edges in $G$ share a vertex. More formally, $ef \in E'$ if there exists $u, v, w \in V$ such that $e = uv$ and $f = uw$.

Prove that the line graph of $K^n$ has chromatic number either $n - 1$ or $n$.

Prove that for odd $n$ the answer is $n$, and for even $n$ the answer is $n - 1$.

6. * Let $G = (V, E)$ denote a graph. Let $V_0 = \{u' \mid u \in V\}$, so that $V_0 \cap V = \emptyset$ (think of $V_0$ as of a copy of $V$). Using $G$ we construct the graph $G' = (V', E')$ as follows: $V' = V \cup V_0 \cup \{z\}$, $z \not\in V \cup V_0$, $E' = E \cup \{u'v \mid uv \in E\} \cup \{zu' \mid u' \in V_0\}$. Show that $\chi(G') = \chi(G) + 1$. 