Graph Theory: Problem set 5

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1. Let \( \kappa(G) \) denote the minimum size of a vertex set \( S \) such that \( G - S \) is disconnected or contains only 1 vertex. Let \( \kappa'(G) \) denote the minimum size of an edge set \( T \) such that \( G - T \) has more than one component. Show that

\[
\kappa(G) \leq \kappa'(G) \leq \delta(G)
\]

where \( \delta(G) \) denotes the minimum degree of \( G \).

2. If \( G \) is a 3-regular graph, then show that \( \kappa(G) = \kappa'(G) \).

3. Show that every planar graph with \( n \) vertices which has no triangular face has at most \( 2n - 4 \) edges.

4. Let us denote by \( v_i \) the number of degree \( i \) vertices in a planar graph \( G \) on at least 3 vertices. Prove the following inequality

\[
12 \leq \sum_{i=1}^{\infty} (6 - i)v_i.
\]

5. Show that the following are equivalent for a plane graph \( G \).

(a) \( G \) is bipartite.
(b) Every face has even length.
(c) the dual graph \( G^* \) is Eulerian.

6. * Let \( G \) be a plane graph on \( n \) vertices, which has no face shorter than 4 and no two faces of length at most 5 that share an edge. Prove that \( G \) can have at most \( \frac{12}{7}n \) edges.

7. We call the planar graph outerplanar if it can be embedded into \( \mathbb{R}^2 \) in a way that all of its vertices lie on the outer face.

Show that an outerplanar graph on \( n > 2 \) vertices can contain at most \( 2n - 3 \) edges.