Graph theory - problem set 5

March 22, 2018

Exercises

1. Determine if the following graphs are planar or not.

2. Determine all positive integers $r$ and $s$ for which $K_{r,s}$ is planar.

3. Let $G$ be a graph on $n$ vertices and $3n - 6 + k$ edges for some $k > 0$. Then any drawing of $G$ in the plane contains at least $k$ crossing pairs of edges.

4. Let $G$ be a planar graph with fewer than 12 vertices. Show that $G$ has a vertex of degree at most 4.

Problems

5. Show, using Euler’s formula, that if $G$ is a planar graph on $n$ vertices that has finite girth $g$, then $G$ has at most $\frac{g}{2-2}(n-2)$ edges. Deduce that the Petersen graph is not planar.

6. (a) Let $G$ be a planar graph containing no triangles. Show that $\chi(G) \leq 4$.
   
   (b) Let $G$ be a planar graph containing at most three triangles. Show that $\chi(G) \leq 4$.

7. Prove that for any three vertices $x, y, z$ of a planar graph on $n$ vertices, the sum of the degrees $d(x) + d(y) + d(z)$ is at most $2n + 2$.

8. Let $S$ be a set of $n$ points in the plane such that any two of them have distance at least 1. Show that there are at most $3n - 6$ pairs of distance exactly 1.

   [Hint: utilauqeni elgnairt eht gnissu gnissorc on sah hparg eht evorP]

9. Show that every graph can be embedded (drawn) in $\mathbb{R}^3$ with straight-line edges.

10. Let $G$ be a plane graph with triangular faces, and suppose the vertices are colored arbitrarily with three colors. Prove that there is an even number of triangles that get all three colors.