Graph Theory: Problem set 3

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1. a) Every 1-connected graph, which is not 2-connected, is a tree.
   False, consider a cycle with an additional edge attached to it.

   b) Every tree is 1-connected and not 2-connected.
   True. Any neighbor of a leaf is a cut vertex, if the tree has at least three vertices. Possibly a tree consisting only of one edge could be a counterexample. However, in general if a graph has less than k+1 vertices, k-connectedness is usually not defined.

   c) No k-connected graph can contain a vertex of degree k – 1.
   True. The neighbors of a vertex v of degree k – 1 separate it from the rest of the graph. Again, we do not define k-connectedness for a graph on less than k+1 vertices.

   d) Every graph with minimal vertex degree k is k-connected.
   False. A cut-vertex can have an arbitrary degree.

2. Prove without using Menger’s theorem that in a 2-connected graph every two vertices lie on a common cycle.

   By using the fact that every 2-connected graph can be constructed by successively adding H-paths to a cycle we prove our claim by induction on the number of edges in G. Thus, the trivial case is when whole G is a cycle, otherwise we can assume that we can obtain G by adding an H-path wPz to a proper 2-connected subgraph H of G. If we pick two arbitrary vertices u and v from V(G), either both of them belong to H, both of them belong to P \ H, or one of them belongs to H, let’s say u, and the other to P \ H. The first case can be handled by induction hypothesis. For the second case it is enough to use the fact that H is connected. Indeed, a path wP_0z within H can be extended with wPz to the cycle containing u and v. For the third case we show that there is a path from w to z via u. By induction hypothesis w and u lie on a common cycle C_1 and z and u lie on a common cycle C_2. If z \in C_1 we are done. Otherwise let f be the first vertex belonging to C_2 on a path P_1 from w to u that takes place in C_1. f is well defined because at least u belongs to C_2. Let fP_2z be the path that takes place in C_2, and contains u (see Figure
1 for an illustration). By the definition of $f wP_1 f P_2 z$ is the path from $w$ to $z$ containing $u$. Putting $wP_1 f P_2 z$ and $wP z$ gives us the cycle containing $u$ and $v$.

![Figure 1: The path $P_1 P_2$](image)

3. Prove that any $k$-regular connected bipartite graph is 2-connected.

We can assume that $k > 1$, as other cases are trivial. We proceed by contradiction. Let $v$ be a cutvertex in a $k$-regular connected bipartite graph $G$ with parts $V_1$ and $V_2$. Let $v \in V_1$. Let $C_1 \ldots C_i$ denote the connected components of $G \setminus v$. Let $k'$ denote the number of neighbors of $v$ in $C_1$. If we count the edges in $C_1$ by looking at the vertices in $V(C_1) \cap V_1$ we get that $k$ divides $|E(C_1)|$, because every vertex in $V(C_1) \cap V_1$ has degree $k$. On the other hand if we count the edges in $C_1$ by looking at the vertices in $V(C_1) \cap V_2$, we get that $k$ does not divide $|E(C_1)|$. Indeed, all vertices in $V(C_1) \cap V_2$, but $k'$, where $0 < k' < k$, have degree $k$, and $k'$ vertices has degree $k - 1$. Hence, in total the number of edges $k'(k-1) + lk = k(k' + l) - k'$, for some $l$, cannot be divisible by $k$ (contradiction).

4. Using Dilworth theorem prove that any sequence of real numbers of length $n^2 + 1$ contains a monotone increasing or a monotone decreasing subsequence of length $n + 1$.

Let $S = (x_1, \ldots, x_{n^2+1})$ denote a sequence of real numbers. We associate with the element $x_i$ the point $X_i = (i, x_i)$ in the plane. Let us define the relation $\leq_r$ on the set of points $X = \{X_i \mid i = 1, \ldots, n^2 + 1\}$ as follows: $(a, b) \leq_r (c, d)$ iff $a \leq c$ and $b \leq d$. It is straightforward to check that $(X, \leq_r)$ is a poset.

Let $m$ denote the length of a longest chain in $X$. If $m \geq n + 1$, we are done, since a chain corresponds to the monotone increasing subsequence of $S$.

If $m < n + 1$, by Dilworth theorem we can partition $X$ into at most $m$ anti-chains. If the length of each of these anti-chains is less than $n + 1$, we obtain a contradiction. Indeed, we partition $X$ into at most $n$ anti-chains each of which has at most $n$ elements. Thus, we could have at most $n \times n$ elements in $S$. However, the size of $S$ is $n^2 + 1$. Hence, we can find an anti-chain of size $n + 1$, which corresponds to the monotone decreasing subsequence of $S$, and that concludes the proof.

5. * Prove marriage (Hall) theorem by using Tutte’s theorem.
Suppose that Hall theorem holds for bipartite graphs having the two parts of the vertices of the same size.

Let $G = (V = A \cup B, E)$ denote a bipartite graph with bipartition $A$ and $B$, so that $A \leq B$. Suppose that marriage condition holds for $A$, i.e. $|N(A')| \geq A'$, for any $A' \subseteq A$.

First, we reduce the problem to the situation when $|A| = |B|$. This can be done by adding dummy vertices to $A$ and connect each of them with every vertex in $B$.

In what follows we show that if $|A| = |B|$ and the marriage condition is satisfied, then for $G$ the condition of Tutte’s theorem holds as well. Let $S \subseteq V$ denote a non-empty subset of $V$. Let $S_A = A \cap S$. Let $c_B$ denote the number of connected components of $G \setminus S$ having more vertices of $B$ than $A$. Analogously, we define $c_A$.

We claim that $c_B \leq |S_A|$. Indeed, let $B'$ denote the subset of $B$ consisting of the vertices in the union of these connected components. By marriage condition we have at most $|S_A|$ of such components, since the number of neighbors of the vertices in $B'$ in $S \setminus S_A$ is at most $|B'| - c_B$. Similarly, we prove that $c_A \leq |S_B|$.

Let $c$ denote the number of odd connected components in $G \setminus S$. Obviously, $c_A + c_B = c$. Thus, $c_A + c_B = c \leq |S|$. Hence, the Tutte’s condition is satisfied, and marriage theorem follows.

6. **Hint:** Define a partial order on the vertices by directing all edges from $X$ to $Y$ for $V = X \cup Y$. By Dilworth’s theorem, one can partition the vertices into $k$ chains, such that the size of the largest antichain $A$ is $k$. Argue that the number of chains of length two (in our partition) equals the size of the complement of $A$. 