Graph theory - problem set 3
March 9, 2017

Exercises

1. Find a minimum vertex cover in the following graph.

2. In a graph $G$ on $n$ vertices, let $\tau(G)$ denote the minimum size of a vertex cover, $\nu(G)$ denote the maximum size of a matching, and let $\alpha(G)$ be the maximum size of an independent vertex set (a set $S \subseteq V(G)$ is independent if $G$ contains no edge with both endpoints in $S$).

(a) Show that $\tau(G) + \alpha(G) = n$.

(b) Show that if $G$ is bipartite then $\nu(G) + \alpha(G) = n$.

3. Let $G$ be a bipartite graph on $2n$ vertices such that $\alpha(G) = n$.

(a) Show that both parts of $G$ contain $n$ vertices.

(b) Check that Hall’s condition holds for $G$ and then deduce that $G$ has a perfect matching.

4. Find a stable matching in the bipartite graph with parts $\{1, 2, 3, 4\}$ and $\{x, y, z, t\}$ and preferences:

$$
\begin{align*}
1 & : z \ y \ x \ t \\
2 & : x \ y \ t \ z \\
3 & : x \ t \ z \ y \\
4 & : x \ t \ y \ z \\
x & : 2 \ 1 \ 3 \ 4 \\
y & : 3 \ 4 \ 1 \ 2 \\
z & : 2 \ 3 \ 4 \ 1 \\
t & : 1 \ 4 \ 3 \ 2
\end{align*}
$$

5. Prove that the matching $S^*$ that Gale-Shapley algorithm outputs (on $G$ with parts $A \cup B$ when elements of $A$ propose) gives the worst possible match for every element in $B$ in a stable matching.

Problems

6. (a) Prove that every $k$-regular bipartite graph (with $k \geq 1$) has a perfect matching.

(b) Construct a 3-regular graph without a perfect matching.

7. [defect version of Hall’s theorem]

Show that if $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for every $S \subseteq A$, then $G$ has a matching with at least $|A| - d$ edges.

This $d$ is called the defect of $S$. Find a set in the graph of Exercise 1 with maximum defect.

8. Is it true that for a given set of preferences, the Gale-Shapley algorithm always produces the same number of “proposals” (no matter what order of proposals it chooses)?

9. Prove that any bipartite graph $G$ has a matching of size at least $|E(G)|/\Delta(G)$.

10. An $r \times s$ Latin rectangle is an $r \times s$ matrix $A$ with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a Latin square. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

11. Let $X$ be a finite set and let $\mathcal{S}$ be a set of subsets of $X$. A system of distinct representatives is an injective map $\varphi : \mathcal{S} \to X$ such that for each $S \in \mathcal{S}$ we have $\varphi(S) \in S$. Give a necessary and sufficient condition for $\mathcal{S}$ to have a system of distinct representatives.

12. Let $G$ be a graph on $2n$ vertices, such that all degrees are at least $n$. Show that $G$ has a perfect matching.