Exercises

1. Let $T$ be a tree and $e$ be an edge of $T$. Prove that $T - e$ is not connected.

2. Let $T$ be a tree and let $u$ and $v$ be two non-adjacent vertices of $T$. Prove that $T + uv$ contains a unique cycle.

3. Show that a graph is connected if and only if it contains a spanning tree.

4. Prove that a forest on $n$ vertices with $c$ connected components has exactly $n - c$ edges.

5. Find a maximum matching in the following graph.

6. As we have seen in class, if $G$ is a bipartite graph and $M$ is a matching in $G$ that is not maximum (i.e. $G$ contains a larger matching) then the graph contains an augmenting path. Is this always true if $G$ is not bipartite?

Problems

7. Let $T$ be a tree on $n$ vertices that has no vertex of degree 2. Show that $T$ has more than $n/2$ leaves.

8. Show that every tree $T$ has at least $\Delta(T)$ leaves.

9. Prove that in a tree, there is at most one perfect matching.

10. Show that a graph $G$ contains at least $|E(G)| - |V(G)| + 1$ cycles.

11. Let $T$ be an $n$-vertex tree that has exactly $2k$ vertices of odd degree. Show that $T$ can be split into $k$ edge-disjoint paths (i.e., $T$ is the union of $k$ edge-disjoint paths).

12. Let $T$ be a tree on $t$ vertices and suppose $G$ is a graph with $\delta(G) \geq t - 1$. Show that $T \subseteq G$, i.e., $G$ has a subgraph isomorphic to $T$.

13. Prove that a connected graph $G$ is a tree if and only if any three pairwise (vertex-)intersecting paths in $G$ have a common vertex.