Graph Theory: Problem set 13

May 30, 2011

1. Find the maximum $s - t$ flow in the following network using the algorithm from the lecture.

![Network Diagram](image1)

Figure 1: Network

![Flow Diagram](image2)

Figure 2: Flow $f$ of total value 11

The following represents a possible sequence of steps carried out by our algorithm. Note that based on the strategy according to which we traverse the graph the sequence of steps and hence also resulting the flow might look differently. However, the total value of the flow in the end should always be the same.

The algorithm first constructs a flow $f_1$ along the path $sv_1v_2v_3t$ with value 3. $f_1$ is augmented into $f_2$ by sending a flow with the value of 2 along the path $sv_1v_2v_4v_5t$. Then we get from $f_2$ the flow $f_3$ by sending a flow with the value of 3 along the path $sv_6v_7v_8v_3t$. The last improvement can be made by sending a flow with the value of 3 along the path $sv_9v_{10}v_5t$. Thereby we have obtained the flow $f$ (see Figure 6 for an illustration). The vertices marked by a spiral correspond to the marked vertices after the last round.

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2. Find the maximum $s - t$ flow in the following networks using the algorithm from the lecture.

The following represents a possible sequence of steps carried out by our algorithm. Note that based on the strategy according to which we traverse the graph the sequence of steps and hence also the resulting the flow might look differently. However, the total value of flow in the end should always be the same.

Undirected case:
The algorithm first construct a flow $f_1$ along the path $suvt$ with value 3. $f_1$ is augmented into $f_2$ by sending a flow with value 3 along the path $swvt$. Then we get from $f_2$ the flow $f_3$ by sending a flow with value 2 along the path $suzt$. Finally, we have obtained the flow $f$ (see Figure 5 for an illustration) by sending a flow with value 1 along the path $swzvt$. The vertices marked by a spiral correspond to the marked vertices after the last round.

Directed case:
The algorithm first construct a flow $f_1$ along the path $suwzt$ with value 2. $f_1$ is augmented into $f_2$ by sending a flow with value 2 along the path $swvt$. Then we get from $f_2$ the flow $f_3$ by sending a flow with value 2 along the path $suzvt$. Finally, we have obtained the flow $f$ (see Figure 6 for an illustration) by sending a flow with value 1 along...
the path swut. The vertices marked by a spiral correspond to the marked vertices after the last round.

3. Formulate and prove Ford and Fulkerson’s Theorem for the networks containing \(n\) sources and \(m\) sinks (using Ford and Fulkerson’s Theorem).

Let \(N\) be a network with underlying graph \(G = (V, E)\), and let \(V_s \subseteq V\) and \(V_t \subseteq V\) be the sets containing sources and sinks in \(N\), respectively. The maximum value of the flow between \(V_s\) and \(V_t\) equals to the capacity of minimum edge cut separating \(V_s\) and \(V_t\). By a cut separating \(V_s\) and \(V_t\) we understand a set of edges whose removal disconnect the graph such that no source and sink are in the same connected component.

The proof follows easily from Ford and Fulkerson’s Theorem.

We introduce two new vertices to \(G\), namely \(s\) and \(t\), which will play a role of the super-source and super-sink, resp. Then we join each vertex of \(V_s\) to \(s\) by an edge directed from \(s\), and each vertex in \(V_t\) to \(t\) by an edge directed towards \(t\). On the edge \(ss', s \in V_s\), we put the capacity that equals to the sum of the capacities of the edges incident to \(s'\) in \(G\). Analogously, we define the capacities for the edges incident to \(t\) (see Figure 7 for an illustration). Let us call resulting network \(N'\), and its underlying graph \(G' = (V', E')\). Now, we can apply Ford and Fulkerson’s Theorem on \(N'\). Every \(s - t\) flow in \(N'\) corresponds to \(V_s - V_t\) flow in \(N'\) having the same value. Moreover, it is easy to see that the capacity of the minimum cut separating \(V_s\) and \(V_t\) in \(N\) equals to the capacity of the minimum cut separating \(s\) and \(t\) in \(N'\), because in fact every such cut in \(N'\) correspond to a cut in \(N\) having the same capacity.

![Figure 6: Flow \(f\) of total value 7](image)

We prove the above claim. Let \(E_0 \subseteq E\) denote a cut in \(N\) separating \(V_s\) from \(V_t\). If after
removing $E_0$ from $E$, there is a path between $s$ and $t$, this path must connect a vertex in $V_s$ with a vertex in $V_t$. Thus, every cut in $N$ is a cut in $N'$.

On the other hand given a minimal cut $E_0$ in $N'$ we can construct a cut $E_1$ in $N$ as follows $E_1 = (E_0 \setminus (\{su | su \in E'\} \cup \{vt | vt \in E'\})) \cup \{uv | u \in V_s \text{ and } su \in E_0\} \cup \{vu | u \in V_t \text{ and } ut \in E_0\}$. This means that for each edge $e$ not in $E$ we introduce the surrogate edges from $E$ having in total the same capacity as $e$. Moreover, a minimal cut in $G'$ cannot contain both $sv$ and $vu$, for some $v \in V_s$ and $u \neq s$, or both $vt$ and $uv$, for some $v \in V_t$ and $u \neq t$. If that were the case, by the removal of $uv$ from our cut, we would still have a cut (contradicting minimality of $E_0$). Thus, $E_0$ and $E_1$ have the same capacity. Here, in fact we were trying to prove only that the capacity of $E_1$ cannot be smaller than the capacity of $E_0$, as the other direction was true by definition.

Finally, we obtain that we have an $s-t$ flow $f$ in $N'$ with the value of the capacity of a minimal cut in $N'$. Luckily $f$, if restricted to $E$, is in fact $V_s - V_t$ flow in $N$ with the same value.

4. Let $N$ be a network with sink $s$ and source $t$. Let $f$ be a flow in $N$. Let $G = (V, E)$ be the underlying graph of $N$. If $S \subseteq V$, then we denote by $\overline{S}$ the complement of $S$ in $V$. Let $S \subseteq V$ such that $s \in S$ and $t \notin S$. Prove that every cut $(S, \overline{S})$ in $N$ satisfies $f(S, \overline{S}) = f(s, V)$ by induction on the number of nodes in $S$.

The base case when $S = \{s\}$ trivial since it says that $f(s, V) = f(\{s\}, V)$. For the inductive case observe that for any vertex $v \in S$ the following is true.

$$-\sum_{s \in S} f(\overline{v}s) + \sum_{s \in S} f(\overline{s}v) = \sum_{s \in \overline{S} \setminus v} f(\overline{v}s) - \sum_{s \in \overline{S} \setminus v} f(\overline{s}v) \quad (1)$$

where by $\overline{v}s$ and $\overline{s}v$ we have denoted the edge oriented from $v$ and towards $v$, respectively. Indeed, the above equality is just rewritten Kirchhoff’s Law:

$$\sum_{s \in S \setminus \overline{v}} f(\overline{s}v) + \sum_{s \in S} f(\overline{s}v) = \sum_{s \in \overline{S} \setminus v} f(\overline{v}s) + \sum_{s \in S} f(\overline{s}v)$$

$$\sum_{s \in V \setminus v} f(\overline{s}v) = \sum_{s \in S \setminus \overline{v}} f(\overline{s}v)$$

By adding $v$ to $S$ we decrease $f(S, \overline{S})$ by $\sum_{s \in S} f(\overline{s}v) - \sum_{s \in S} f(\overline{v}s)$ and increase by $\sum_{s \in \overline{S} \setminus v} f(\overline{v}s) - \sum_{s \in \overline{S} \setminus v} f(\overline{s}v)$ (see Figure 8 for an illustration). Luckily these both values are the same by (1). Thus, in the end $f(S, \overline{S})$ stays unchanged, which concludes the proof.
Figure 8: Addition of a vertex $v$ to $S$