Graph Theory: Problem set 11

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1. By Mantel’s Theorem any triangle-free graph $G$ contains at most $\alpha(G)\tau(G)$ edges, where $\alpha(G)$ and $\tau(G)$, respectively, stand for the maximum size of an independent set and minimum size of a vertex cover in $G$, respectively.

Show that the only graph on $n$ vertices without a triangle, that has $\lfloor n^2/4 \rfloor$ edges, is the complete balanced bipartite graph.

2. For $0 < s \leq t \leq n$ let $z(n, s, t)$ denote the maximum number of edges in a bipartite graph whose partition sets both have size $n$, and which does not contain $K_{s,t}$. Show that $2ex(n, K_{s,t}) \leq z(n, s, t) \leq ex(2n, K_{s,t})$.

3. Let $G$ denote a graph on $n$ vertices with chromatic number $k$. At most how many edges can $G$ have?

4. Let $H = (V, E)$, $n' = |V|$, denote a graph on at least 2 vertices.

Prove that the maximal number of edges a graph on $n$ vertices can have, if it does not contain two disjoint copies of $H$ as a subgraph, is at most $ex(n, H) + nn' + c_H$, where $c_H$ is a constant depending only on the number of vertices in $H$.

5. Let $T$ denote a tree on 4 vertices. Prove that if $G$ has minimum degree 3, then $T$ is a subgraph of $G$.

6. Let $n, m, s$ and $t$ denote four natural numbers such that $0 < s < n$ and $0 < t < m$.

Show that by deleting at most $(n - s)(m - t)/s$ edges from $K_{n,m}$ we cannot destroy all its $K_{s,t}$ subgraphs.