Graph theory - problem set 11
May 17, 2018

Exercises

1. Let \((\Omega, P)\) be a probability space. Prove that for any collection of events \(E_1, \ldots, E_k\), we have

\[ P\left[ \bigcup_{i=1}^{k} E_i \right] \leq \sum_{i=1}^{k} P[E_i], \]

and if \(E_1, \ldots, E_k\) are disjoint events, then we have equality here.

2. Let \(\sigma\) be an arbitrary permutation of \(\{1, \ldots, n\}\), selected uniformly at random from the set of all permutations (that is, each permutation is selected with probability \(\frac{1}{n!}\)). What is the expectation of the number of fixed points in \(\sigma\)? (Recall that \(i\) is a fixed point if \(\sigma(i) = i\).)

3. Take a complete graph \(K_n\) where each edge is independently colored red, green or blue with probability \(1/3\). What is the expected number of red cliques of size \(a\) in this graph?

Problems

4. Suppose \(r \geq 4\) and let \(H\) be an \(r\)-uniform hypergraph with at most \(4r^{-1}/3^r\) edges. Prove that there is a coloring of the vertices of \(H\) by four colors so that in every edge all four colors appear.

5. Let \(G\) be a graph with \(m\) edges, and let \(k\) be a positive integer. Prove that the vertices of \(G\) can be colored with \(k\) colors in such a way that there are at most \(m/k\) monochromatic edges (i.e., edges with both endpoints colored the same).

6. Prove that if \(G\) has \(2n\) vertices and \(e\) edges then it contains a bipartite subgraph with at least \(e\frac{n}{2n-1}\) edges. [Use a random partition of the vertices into two parts of size \(n\)]

7. (a) Let \(G\) be a \(K_{1,d}\)-saturated graph. Prove that the vertices of \(G\) whose degrees are smaller than \(d - 1\) form a clique.

(b) Use this to prove that \(\text{sat}(n, K_{1,d}) \geq \frac{n(d-1)}{2} - \frac{d^2}{8}\).

(c) Prove that \(\text{sat}(n, K_{1,d}) = \frac{n(d-1)}{2} - \frac{d^2}{8}\) if \(d\) is even and \(n - d/2\) is also even.

[This problem is not probabilistic]

8.* In an \(n \times n\) matrix, each of the numbers \(1, 2, \ldots, n\) appears exactly \(n\) times. Show that there is a row or a column in the matrix with at least \(\sqrt{n}\) distinct numbers.