Graph theory - problem set 10

May 4, 2017

Exercises

1. Determine the chromatic number of the first graph and the edge-chromatic number of the second graph below.

2. Let $G$ be a graph on $V$, and let $G[X]$ be the subgraph induced by $X \subseteq V$ (i.e. the graph with vertex set $X$ containing the edges of $G$ with both endpoints in $V$). Prove that $\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$.

3. Are the following statements true?
   (a) If $G$ and $H$ are graphs on the same vertex set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$.
   (b) Every graph $G$ has a coloring with $\chi(G)$ colors where $\alpha(G)$ vertices get the same color.

4. (a) A fair coin is tossed 100 times. What is the expected number of tails? (A fair coin comes up heads or tails with probability 1/2 each.)
   (b) A fair die is thrown 5 times. Let $X$ be the sum of the 5 obtained values. What is bigger, $P(X \leq 5)$ or $P(X \geq 30)$? (A fair die has 6 faces with values 1,...,6 obtained with equal probability.)
   (c) A fair die is thrown 2 times. Calculate the probability that the sum of the values is odd.

Problems

5. Let $G$ be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs of distinct vertices $x, y \in V(G)$. Prove that $G$ is the complete graph.

6. (a) Find the edge-chromatic number of $K_{2n+1}$ (don’t use Vizing’s theorem).
    (b) Find the edge-chromatic number of $K_{2n}$.

7. Let $G$ be a graph on $n$ vertices and $\overline{G}$ be its complement. Prove that
   (a) $\chi(G)\chi(\overline{G}) \geq n$.
   (b) $\chi(G) + \chi(\overline{G}) \leq n + 1$.

8. Let $G = (A, B; E)$ be bipartite with maximum degree $\Delta$. Prove that $\chi_e(G) = \Delta$.
   Remark: This theorem of König strengthens Vizing’s theorem for bipartite graphs.