Graph theory - problem set 1
February 23, 2017

Exercises

1. (a) Is $C_n$ a subgraph of $K_n$?
   (b) For what values of $n$ and $m$ is $K_{n,n}$ a subgraph of $K_m$?
   (c) For what $n$ is $C_n$ a subgraph of $K_{n,n}$?

2. Given a graph $G$ with vertex set $V = \{v_1, \ldots, v_n\}$ we define the degree sequence of $G$ to be the list $d(v_1), \ldots, d(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:
   (a) 3, 3, 2, 2, 1
   (b) 6, 6, 6, 4, 4, 3, 3

3. Construct two graphs that have the same degree sequence but are not isomorphic.

4. A graph is $k$-regular if every vertex has degree $k$. Describe all 1-regular graphs and all 2-regular graphs.

5. Prove that the number of odd-degree vertices in a graph is always even.

6. How many (labelled) graphs exist on a given set of $n$ vertices? How many of them contain exactly $m$ edges?

Problems

7. Do graphs with the following degree sequences exist:
   (a) 6, 6, 6, 4, 4, 2, 2
   (b) 6, 6, 6, 6, 5, 4, 2, 1?

8. Let $G$ be a graph with minimum degree $\delta > 1$. Prove that $G$ contains a cycle of length at least $\delta + 1$.

9. How many (labelled) graphs on the vertex set $\{1, \ldots, n\}$ are isomorphic to $P_n$? How many are isomorphic to $C_n$?

10. Show that every graph on at least two vertices contains two vertices of equal degree.

11. For every $n \geq 6$, find a graph on $n$ vertices, and $n + 2$ edges that contains exactly 6 cycles.

12. What is the maximum number of edges in a bipartite graph on $n$ vertices? (Prove your answer.)

13. (a) Let $G$ be a graph containing a cycle $C$, and assume that $G$ contains a path of length at least $k$ between two vertices of $C$. Show that $G$ contains a cycle of length at least $\sqrt{k}$.
   *(b) Prove the same statement with $\sqrt{2k}$ instead of $\sqrt{k}$. 