Packing and covering - problem set 3
March 4, 2014.

1. Let $P$ be a convex polygon contained in a convex polygon $Q$. Show that $\text{Per}(P) \leq \text{Per}(Q)$.

2. Let $C$ be a convex set and $p_n$ be an $n$-gon with largest area inscribed in $C$. Show that $\frac{A(C) - A(p_n)}{A(C)} \leq \frac{c}{n^2}$ for some fixed constant $c$.

3. Euler’s formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections, and $v$ is the number of vertices, $e$ is the number of edges and $f$ is the number of faces (regions bounded by edges, including the outer, infinitely large region), then $v-e+f=2$. Show that if $G$ is a planar graph with $n$ vertices, then it has at most $3n-6$ edges and that, evenmore, if $G$ is bipartite then it has at most $2n-4$ edges.

4. A convex hexagon which is the image of a regular hexagon under an affine transformation is said to be *affinely regular*. Show that a convex hexagon $p_1, \ldots, p_6$ is affinely regular if and only if it is centrally symmetric and $\vec{p}_2 \vec{p}_1 + \vec{p}_2 \vec{p}_3 = \vec{p}_3 \vec{p}_4$.

5*. Suppose you have three boxes. The first one contains two one-dollar coins, the second one one-dollar coin and one-franc coin, and the third two one-franc coins. One of these boxes is chosen at random, and from it one of the coins is taken out at random. This coin turns out to be a one-dollar coin. What is the probability that the remaining coin, in the chosen box, is also a one-dollar coin?

6. (Optional exercise) The Gamma function is defined by $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$. It satisfies the following

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(2) = 1, \quad \Gamma(x+1) = x\Gamma(x), \quad \forall x > 0.$$  

Using integration by parts, prove that

$$n \cdot \int_0^{\pi/2} \cos^n \alpha \, d\alpha = (n-1) \int_0^{\pi/2} \cos^{n-2} \alpha \, d\alpha.$$

By induction on $n$, show that

$$\int_0^{\pi/2} \cos^n \alpha \, d\alpha = \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2} + 1\right)}.$$

Let $\kappa_d$ denote the volume of the unit ball in $\mathbb{R}^d$. Show that

$$\kappa_d = \kappa_{d-1} \int_{-1}^{1} (\sqrt{1-x^2})^{d-1}dx.$$

Using the above formulas, eventually prove by induction on $d$ that

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}.$$