Incidences between planes over finite fields

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Abstract

In this note, we use methods from spectral graph theory to obtain bounds on the number of incidences between $k$-planes and $h$-planes in $\mathbb{F}_q^d$ which generalize a recent result given by Bennett, Iosevich, and Pakianathan (2014). More precisely, we prove that the number of incidences between a set $K$ of $k$-planes and a set $H$ of $h$-planes with $h \geq 2k + 1$, which is denoted by $I(K, H)$, satisfies

$$\left| I(K, H) - |K||H|q^{(d-h)(k+1)} \right| \lesssim q^{(d-h)h+k(2h-d-k+1)/2} \sqrt{|K||H|}.$$ 

As an application of incidence bounds, we prove that almost all $k$-planes, $1 \leq k \leq d-1$, are spanned by a set of $3q^{d-1}$ points in $\mathbb{F}_q^d$. We also obtain results on the number of $t$-rich incidences $k$-planes and $h$-planes in $\mathbb{F}_q^d$, with $t \geq 2$.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of $q$ elements where $q$ is an odd prime power. Let $\mathcal{P}$ be a set of points, $\mathcal{L}$ be a set of lines, and $I(\mathcal{P}, \mathcal{L})$ be the number of incidences between $\mathcal{P}$ and $\mathcal{L}$. Bourgain, Katz, and Tao [2] proved that the number of incidences between a set of $N$ points and a set of $N$ lines is at most $O(N^{3/2-\epsilon})$. Note that one can easily obtain the bound $N^{3/2}$ by using the Turán number and the fact that two lines intersect in at most one point. Here and throughout, $X \gtrsim Y$ means that $X \geq CY$ for some constant $C$ and $X \gg Y$ means that $Y = o(X)$, where $X, Y$ are viewed as functions of the parameter $q$.

The relationship between $\epsilon$ and $\alpha$ in the result of Bourgain, Katz, and Tao is difficult to determine, and it is far from tight. If $N \ll q$, then Grosu [7] proved that one can embed the point set and the line set to $\mathbb{C}^2$ without changing the incidence structure. Thus it follows from a tight bound on the number of incidences between points and lines in $\mathbb{C}^2$ due to Tóth [20] that $I(\mathcal{P}, \mathcal{L}) = O(N^{4/3})$. By using methods from spectral graph theory, the third listed author [22] gave a tight bound for the case $N \gg q$ as follows.

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Theorem 1.1 (Vinh, [22]). Let \( P \) be a set of points and \( L \) be a set of lines in \( \mathbb{F}_q^2 \). Then we have

\[
\left| I(P, L) - \frac{|P||L|}{q} \right| \leq q^{1/2} \sqrt{|P||L|}.
\]

(1.1)

It follows from the upper bound of Theorem 1.1 that if \( |P| = |L| = N \geq q^{3/2} \), then the number of incidences between points and lines is at most \((1 + o(1))N^{4/3}\), which meets the Szemerédi-Trotter bound. Theorem 1.1 has many interesting applications in several combinatorial problems, see for example [8, 9, 13, 22].

It also follows from the lower bound that if \(|P||L| \gtrsim q^3\), then there exists at least one pair \((p, l) \in P \times L\) such that \(p \in l\). The lower bound of Theorem 1.1 is proved to be sharp up to a constant in the sense that there exist a point set \( P \) and a line set \( L \) with \(|P| = |L| = q^{3/2}\) without incidences (see [23] for more details). Furthermore, the third listed author proved that almost every point set \( P \) and line set \( L \) in \( \mathbb{F}_q^d \) of cardinality \(|P| = |L| \gtrsim q\), there exists at least one incidence \((p, l) \in P \times L\). More precisely, the statement is as follows.

Theorem 1.2 (Vinh [23]). For any \( \alpha > 0 \), there exist an integer \( q_0 = q(\alpha) \) and a number \( C_\alpha > 0 \) satisfying the following property. When a point set \( P \) and a line set \( L \) with \(|P| = |L| = s \geq C_\alpha q\), are chosen randomly in \( \mathbb{F}_q^d \), then the probability of \( \{(p, l) \in P \times L : p \in l\} = \emptyset \) is at most \( \alpha^s \), provided that \( q \geq q_0 \).

The third listed author also generalized Theorem 1.1 to the case of points and hyperplanes in \( \mathbb{F}_q^d \) in [22] as follows.

Theorem 1.3 (Vinh [22]). Let \( P \) be a set of points and \( H \) be a set of hyperplanes in \( \mathbb{F}_q^d \). Then the number of incidences between points and hyperplanes satisfies

\[
\left| I(P, H) - \frac{|P||H|}{q^d} \right| \leq (1 + o(1))q^{(d-k)/2} \sqrt{|P||H|}.
\]

By combining the upper bound of Theorem 1.3 and counting arguments, Bennett, Iosevich, and Pakianathan [1] extended Theorem 1.3 to the incidences between points and \( k \)-planes, where a \( k \)-plane is defined as follows.

Definition 1.4. Let \( V \) be a subset in \( \mathbb{F}_q^d \). We say that \( V \) is a \( k \)-plane in \( \mathbb{F}_q^d \) \((k < d)\) if there exist \( k + 1 \) vectors \( v_1, \ldots, v_{k+1} \) in \( \mathbb{F}_q^d \) satisfying

\[
V = \text{span}\{v_1, \ldots, v_k\} + v_{k+1}, \text{ and rank}\{v_1, \ldots, v_k\} = k.
\]

Theorem 1.5 (Bennett et al. [1]). Let \( P \) be a set of points and \( H \) be a set of \( k \)-planes in \( \mathbb{F}_q^d \). Then there is no more than

\[
\frac{|P||H|}{q^{d-k}} + (1 + o(1))q^{(d-k)/2} \sqrt{|P||H|}
\]

incidences between \( P \) and \( H \).
In this note, we will extend Theorem 1.5 to the case of $k$-planes and $h$-planes with $h \geq 2k + 1$ in the following theorem.

**Theorem 1.6.** Let $\mathcal{K}$ be a set of $k$-planes and $\mathcal{H}$ be a set of $h$-planes in $\mathbb{F}_q^d$ with $h \geq 2k + 1$. Then the number of incidences between $\mathcal{K}$ and $\mathcal{H}$ satisfies

$$|I(\mathcal{K}, \mathcal{H})| \leq \sqrt{ck} \left(1 + o(1)\right)q^{\frac{(d-h)h + k(2h - d - k + 1)}{2}} |\mathcal{K}| |\mathcal{H}|,$$

where $c_k = (2k + 1)\left(\begin{array}{c} k \\ k/2 \end{array}\right)$.

It follows from Theorem 1.6 that if $|\mathcal{K}| |\mathcal{H}| \geq q^{d(k+h)+2d+k}/q^{k^2+h^2+2h}$, then the set of incidences between $\mathcal{K}$ and $\mathcal{H}$ is nonempty, and if $|\mathcal{K}| |\mathcal{H}| \gg q^{d(k+h)+2d+k}/q^{k^2+h^2+2h}$, then $I(\mathcal{K}, \mathcal{H})$ is close to the expected number $|\mathcal{K}| |\mathcal{H}|/q^{(d-k)(k+1)}$.

As a consequence of Theorem 1.6, we obtain the following result on the number of distinct $k$-planes determined by a set of points in $\mathbb{F}_q^d$ with $1 \leq k \leq d - 1$.

**Theorem 1.7.** Let $\mathcal{P}$ be a set of points in $\mathbb{F}_q^d$. If $|\mathcal{P}| \geq 3q^{d-1}$, then the number of distinct $k$-planes determined by $\mathcal{P}$ is at least $(1 - o(1))q^{(d-k)(k+1)}$.

Theorem 1.7 is sharp up to a constant factor since there is only a hyperplane determined by a set of all points in a hyperplane in $\mathbb{F}_q^d$.

The study of incidence problems and applications over finite fields received considerable amount of attention in recent years, see for example [3, 5, 7, 10, 14, 15, 17, 18, 16, 21].

A related question that has recently received attention is the following: Given a point set $\mathcal{P}$ in $\mathbb{F}_q^2$, what is the cardinality of the set of $k$-rich lines, i.e. lines contain at least $k$ points from $\mathcal{P}$? Note that this question is quite different from the real case. In the real case, it follows from the Szemerédi-Trotter theorem that the number of $k$-rich lines determined by a set of $n$ points is $O(n^2/k^3)$ for any $k \geq 2$, but in the finite fields case, if $k$ is large enough as a function of $|\mathcal{P}|$ and $q$, then it follows from Theorem 1.1 that the number of $k$-rich lines determined by a point set $\mathcal{P}$ is $O(q|\mathcal{P}|/k^2)$ with $k > 2|\mathcal{P}|/q$. Thus, if $k < 2|\mathcal{P}|/q$, we do not obtain any bound from Theorem 1.1.

In [16], Lund and Saraf introduced an approach to fill in this gap. More precisely, they proved that, for any $k \geq 2$, there are at least $cq^2$ $k$-rich lines determined by a point set of cardinality $2(k-1)q$ for some constant $0 < c < 1$. This implies that there are at least $cq^2$ distinct lines determined by a set of $2q$ points. They also proved that

**Theorem 1.8 (Lund-Saraf [16]).** For any integer $t \geq 2$, let $\mathcal{H}$ be a set of the $h$-planes in $\mathbb{F}_q^d$ of the cardinality

$$|\mathcal{H}| \geq 2(t - 1)q^{d-h}.$$

Then the number of points contained in at least $t$ $h$-planes from $\mathcal{H}$ is at least $cq^d$, where $c = (t - 1)/(t - 1 + 2q^{h(d-h-1)})$.

We note that in the case $h < d-1$ and $t < q^{h(d-h-1)}$, the constant $c$ depends on $q$. This condition is necessary since, for instance, one can take a set of $2q^2$ lines in the union of
two planes in $\mathbb{F}_q^3$, then the number of 2-rich points is at most $O(q^2)$. On the other hand, it follows from Theorem 1.8 for the case $d = 3$ and $h = 1$ that the number of 2-rich points is at least $\Omega(q^2)$. This implies that Theorem 1.8 is sharp in this case.

There are some interesting applications of Theorem 1.8 in combinatorial geometry problems, see for example [5, 13, 16, 19].

Using Lund and Saraf’s approach and the properties of plane-incidence graphs in Section 3, we obtain generalizations of their results as follows.

**Theorem 1.9.** For any $t \geq 2$, let $\mathcal{H}$ be a set of $h$-planes in $\mathbb{F}_q^d$ satisfying

$$|\mathcal{H}| \geq 2(t-1)q^{(d-h)(k+1)}.$$  

Then the number of $k$-planes contained in at least $t$ $h$-planes in $\mathcal{H}$ is at least $cq^{(d-h)(k+1)}$, where $c = (t-1)/((t-1) + 2q^{(d-h-1)(h-k)+k})$.

**Theorem 1.10.** For any $t \geq 2$, let $\mathcal{K}$ be a set of $k$-planes in $\mathbb{F}_q^d$ satisfying

$$|\mathcal{K}| \geq 2(t-1)q^{(d-h)(k+1)}.$$  

Then the number of $h$-planes containing at least $t$ $k$-planes in $\mathcal{K}$ is at least $cq^{(d-h)(h+1)}$, where $c = (t-1)/((t-1) + 2q^{(h-k+1)})$.

## 2 Expander Mixing Lemma

We say that a bipartite graph is *biregular* if in both of its two parts, all vertices have the same degree. If $A$ is one of the two parts of a bipartite graph, we write $\text{deg}(A)$ for the common degree of the vertices in $A$. Label the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Note that in a bipartite graph, we have $\lambda_2 = -\lambda_1$. The following variant of the expander mixing lemma is proved in [6]. We include the proof here for the sake of completeness.

**Lemma 2.1 (Expander mixing lemma).** Let $G$ be a bipartite graph with parts $A, B$ such that the vertices in $A$ all have degree $a$ and the vertices in $B$ all have degree $b$. Then, for any two sets $X \subset A$ and $Y \subset B$, the number of edges between $X$ and $Y$, denoted by $e(X, Y)$, satisfies

$$|e(X, Y) - a|B||X||Y| \leq \lambda_3 \sqrt{|X||Y|},$$

where $\lambda_3$ is the third eigenvalue of $G$.

**Proof.** We assume that the vertices of $G$ are labeled from 1 to $|A| + |B|$, and we denote by $M$ the adjacency matrix of $G$ having the form

$$M = \begin{bmatrix} 0 & N \\ N^t & 0 \end{bmatrix},$$

where $N$ is the $|A| \times |B| 0-1$ matrix, with $N_{ij} = 1$ if and only if there is an edge between $i$ and $j$. First, let us recall some properties of the eigenvalues of the matrix $M$. Since
all vertices in $A$ have degree $a$ and all vertices in $B$ have degree $b$, all eigenvalues of $M$ are bounded by $\sqrt{ab}$. Indeed, let us denote the $L_1$ vector norm by $\| \cdot \|_1$, and let $e_v$ be the unit vector having 1 in the position corresponding to vertex $v$ and zeroes elsewhere. One can observe that $\|M^2 \cdot e_v\|_1 \leq ab$, so the absolute value of each eigenvalue of $M$ is bounded by $\sqrt{ab}$. Let $1_X$ denote the column vector of size $|A| + |B|$ having 1s in the positions corresponding to the set of vertices $X$ and 0s elsewhere. Then, we have that

$$M(\sqrt{a}1_A + \sqrt{b}1_B) = b\sqrt{a}1_B + a\sqrt{b}1_A = \sqrt{ab}(\sqrt{a}1_A + \sqrt{b}1_B),$$

$$M(\sqrt{a}1_A - \sqrt{b}1_B) = b\sqrt{a}1_B - a\sqrt{b}1_A = -\sqrt{ab}(\sqrt{a}1_A - \sqrt{b}1_B),$$

which implies that $\lambda_1 = \sqrt{a}b$ and $\lambda_2 = -\sqrt{ab}$ are the first and second eigenvalues, corresponding to the eigenvectors $(\sqrt{a}1_A + \sqrt{b}1_B)$ and $(\sqrt{a}1_A - \sqrt{b}1_B)$.

Let $W^\perp$ be a subspace spanned by the vectors $1_A$ and $1_B$. Since $M$ is a symmetric matrix, the eigenvectors of $M$, except $\sqrt{a}1_A + \sqrt{b}1_B$ and $\sqrt{a}1_A - \sqrt{b}1_B$, span $W$. Therefore, for any $u \in W$, $Mu \in W$, and $\|Mu\| \leq \lambda_3\|u\|$. Let us now remark the following facts:

1. Let $K$ be a matrix of the form $\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$, where $J$ is the $|A| \times |B|$ all-ones matrix. If $u \in W$, then $Ku = 0$ since every row of $K$ is either $1_A^T$ or $1_B^T$.
2. If $w \in W^\perp$, then $(M - (a/|B|)K)w = 0$. Indeed, it follows from the facts that $a|A| = b|B|$, and $M1_A = b1_B = (a/|B|)K1_A$, $M1_B = a1_A = (a/|B|)K1_B$.

Since $e(X,Y) = 1_Y^T M1_X$ and $|X||Y| = 1_Y^T K1_X$,

$$\left|e(X,Y) - \frac{a}{|B|}|X||Y|\right| = 1_Y^T (M - \frac{a}{|B|}K)1_X.$$

For any vector $v$, let $\bar{v}$ be the orthogonal projection onto $W$, so that $\bar{v} \in W$, and $v - \bar{v} \in W^\perp$. Thus

$$1_Y^T (M - \frac{a}{|B|}K)1_X = 1_Y^T (M - \frac{a}{|B|}K)\bar{1}_X = 1_Y^T M\bar{1}_X = \bar{1}_Y^T M\bar{1}_X,$$

so

$$\left|e(X,Y) - \frac{a}{|B|}|X||Y|\right| \leq \lambda_3\|\bar{1}_X\|\|\bar{1}_Y\|.$$

Since

$$\bar{1}_X = 1_X - ((1_X \cdot 1_A)/(1_A \cdot 1_A))1_A = 1_X - (|X|/|A|)1_A,$$

we have $\|\bar{1}_X\| = \sqrt{|X|(1 - |X|/|A|)}$. Similarly, $\|\bar{1}_Y\| = \sqrt{|Y|(1 - |Y|/|B|)}$.

In other words,

$$\left|e(X,Y) - \frac{a}{|B|}|X||Y|\right| \leq \lambda_3\sqrt{|X||Y|(1 - |X|/|A|)(1 - |Y|/|B|)},$$

which completes the proof of the lemma. \qed
Lemma 2.2. Let $G = (A \cup B, E)$ be a biregular graph with $|A| = m$, $|B| = n$. We label vertices of $G$ from 1 to $|A| + |B|$. Let $M$ be the adjacency matrix of $G$ having the form

$$M = \begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix},$$

where $N$ is the $|A| \times |B|$ matrix, and $N_{ij} = 1$ if and only if there is an edge between $i$ and $j$. Let $v_3 = (v_1, \ldots, v_m, u_1, \ldots, u_n)$ be an eigenvector of $M$ corresponding to the eigenvalue $\lambda_3$. Then we have $(v_1, \ldots, v_m)$ is an eigenvector of $NN^T$, and $J(v_1, \ldots, v_m) = 0$, where $J$ is the $m \times m$ all-ones matrix.

Proof. We have

$$M^2 = \begin{bmatrix} NN^T & 0 \\ 0 & N^T N \end{bmatrix}.$$

Since $v_3$ is an eigenvector of $M$ with the eigenvalue $\lambda_3$, $v_3$ is also an eigenvector of $M^2$ with the eigenvalue $\lambda_3^2$. On the other hand,

$$M^2v_3 = \begin{bmatrix} NN^T \cdot (v_1, \ldots, v_m)^T \\ N^T N \cdot (u_1, \ldots, u_n)^T \end{bmatrix} = \lambda_3^2(v_1, \ldots, v_m, u_1, \ldots, u_n)^T.$$

This implies that $(v_1, \ldots, v_m)$ is an eigenvector of $NN^T$ corresponding to the eigenvalue $\lambda_3^2$. We also note that it follows from proof of Lemma 2.1 that if $v_3$ is an eigenvector corresponding the third eigenvalue of $M$, then $Kv_3 = 0$, which implies that $J(v_1, \ldots, v_m) = 0$. Since $G$ is a biregular graph, $\lambda_3^2$ is the second eigenvalue of $NN^T$. □

It follows from Lemma 2.2 that in order to bound the third eigenvalue of $M^2$, it suffices to bound the second eigenvalue of the matrix $NN^T$.

Suppose that $G = (A \cup B, E)$ is a bipartite graph as in Lemma 2.1. For $S_A \subseteq A$, we denote the set of vertices in $B$ that have at least $t$ neighbors in $S_A$ by $R_t(S_A)$. Similarly, we have the definition of $R_t(S_B)$ with $S_B \subseteq B$. In [16], Lund and Saraf obtained the following theorems.

Theorem 2.3 (Lund and Saraf, [16]). For $S_A \subseteq A$, if $|S_A| \geq 2(t-1)|A|/\deg(B)$, then $|R_t(S_A)| \geq c_{t,G}|B|$, where $c_{t,G} = (t-1)/(t-1+2\deg(B)\mu^2)$, $\mu = |\lambda_3|/|\lambda_1|$.

Theorem 2.4 (Lund and Saraf, [16]). For $S_B \subseteq B$, if $|S_B| \geq 2(t-1)|B|/\deg(A)$, then $|R_t(S_B)| \geq c_{t,G}|A|$, where $c_{t,G} = (t-1)/(t-1+2\deg(A)\mu^2)$, $\mu = |\lambda_3|/|\lambda_1|$.

3 Plane-incidence graphs

We now construct the plane-incidence graph $G_P = (A \cup B, E)$ as follows. The first vertex part, $A$, is the set of all $k$-planes, and the second vertex part, $B$, is the set of all $h$-planes. There is an edge between a $k$-plane $K \in A$ and a $h$-plane $H \in B$ if $K$ lies on $H$. It is easy to check that

$$|A| = \frac{q^d(q^d-1)\cdots(q^d-q^k-1)}{q^k(q^k-1)\cdots(q^k-q^h-1)}, \quad |B| = \frac{q^d(q^d-1)\cdots(q^d-q^h-1)}{q^h(q^h-1)\cdots(q^h-q^h-1)}.$$
Now, we will count the degree of each vertex of the graph $G_P$. We first need the following lemmas.

**Lemma 3.1.** Let $K_1 = \text{span}\{u_1, \ldots, u_k\} + u_{k+1}$, and $K_2 = \text{span}\{v_1, \ldots, v_k\} + v_{k+1}$ be two $k$-planes in $\mathbb{F}_q^d$. Then $K_1 \equiv K_2$ if and only if $\text{span}\{u_1, \ldots, u_k\} \equiv \text{span}\{v_1, \ldots, v_k\}$ and $u_{k+1} \in K_2, v_{k+1} \in K_1$.

**Proof.** If $\text{span}\{u_1, \ldots, u_k\} \equiv \text{span}\{v_1, \ldots, v_k\}$ and $u_{k+1} \in K_2, v_{k+1} \in K_1$, then it is easy to check that $K_1 \equiv K_2$. For the inverse case, if $K_1 \equiv K_2$, then $u_{k+1} \in K_2$ and $v_{k+1} \in K_1$. We need to prove that $\text{span}\{u_1, \ldots, u_k\} \equiv \text{span}\{v_1, \ldots, v_k\}$. Indeed, without loss of generality, we assume that there exists an element $u_i$ for some $1 \leq i \leq k$ such that $u_i \notin \text{span}\{v_1, \ldots, v_k\}$, then we will prove that this leads to a contradiction. Since $K_1 \equiv K_2$, $u_i + u_{k+1} \in K_2$. Therefore, there exist elements $a_1, \ldots, a_k \in \mathbb{F}_q$ such that $u_i + u_{k+1} = \sum_{j=1}^k a_j v_j + v_{k+1}$. On the other hand, since $u_{k+1} \in K_2$, there exist elements $b_1, \ldots, b_k \in \mathbb{F}_q$ such that $u_{k+1} = \sum_{j=1}^k b_j v_j + v_{k+1}$. This implies that $u_i = \sum_{j=1}^k (a_j - b_j)v_j$, which leads to a contradiction, and the lemma follows. \qed

**Lemma 3.2.** Let $K_1 = \text{span}\{u_1, \ldots, u_k\} + u_{k+1}$, $K_2 = \text{span}\{v_1, \ldots, v_k\} + v_{k+1}$ be two $k$-planes in $\mathbb{F}_q^d$. For any $h > k$, if the $h$-plane $H = \text{span}\{t_1, \ldots, t_h\} + t_{h+1}$ contains both of them then $H$ can be written as $H = \text{span}\{t_1, \ldots, t_h\} + u_{k+1}$ and $u_1, \ldots, u_k, v_1, \ldots, v_k, u_{k+1} - v_{k+1} \in \text{span}\{t_1, \ldots, t_h\}$.

**Proof.** First we need to prove that for any vector $x \in H$, $H$ can be written as $H = \text{span}\{t_1, \ldots, t_h\} + x$. Indeed, since $x \in H$, $x$ can be presented as $x = \sum_{i=1}^h a_i t_i + t_{h+1}$ with $a_i \in \mathbb{F}_q$. Let $y = \sum_{i=1}^h b_i t_i + t_{h+1}$ be a vector in $H$, then $y$ can also be written as $y = \sum_{i=1}^h (b_i - a_i) t_i + x$. This implies that $y \in \text{span}\{t_1, \ldots, t_h\} + x$. The inverse case $\text{span}\{t_1, \ldots, t_h\} + x \subseteq H$ is trivial.

If $H$ contains both $K_1$ and $K_2$, then $H$ can be presented as $H = \text{span}\{t_1, \ldots, t_h\} + u_{k+1}$ since $u_{k+1} \in H$. It is easy to see that $u_i \in \text{span}\{t_1, \ldots, t_h\}$ for all $1 \leq i \leq k$. Since $K_2$ is contained in $H$, $v_{k+1} \in H$, which implies that $v_{k+1} - u_{k+1} \in \text{span}\{t_1, \ldots, t_h\}$, and $v_i \in \text{span}\{t_1, \ldots, t_h\}$ for all $1 \leq i \leq k$. \qed

**Lemma 3.3.** The degree of each $k$-plane in $A$ is $(1 + o(1))q^{(d-h)(h-k)}$, and the degree of each $h$-plane in $B$ is $(1 + o(1))q^{(h-k)(k+1)}$.

**Proof.** It follows from Lemma 3.1 and Lemma 3.2 that the degree of each $h$-plane in $B$ is

$$\frac{q^h}{q^k} \prod_{i=0}^{k-1} \frac{q^h - q^i}{q^k - q^i} = (1 + o(1))q^{(h-k)(k+1)}.$$ 

In order to count the degree of each $k$-plane in $A$, we will use similar arguments as in the proof of [1, Theorem 2.3]. Let $x(h,k)$ be the number of distinct $k$-planes in a $h$-plane, and $y(h,k)$ the number of distinct $h$-planes in $\mathbb{F}_q^d$ containing a fixed $k$-plane. Then we have

$$y(h,k) = \frac{x(h,k)}{x(d,k)}.$$
Therefore, we just proved that
\[ x(h, k) = \frac{q^h}{q^k} \prod_{i=0}^{k-1} \frac{q^h - q^i}{q^h - q^i}, \]
which implies that
\[ y(h, k) = \prod_{i=k}^{h-1} \frac{q^{d-i} - 1}{q^{k-i} - 1} = (1 + o(1))q^{(d-h)(h-k)}. \]
In short, the degree of each \( k \)-plane in \( A \) is \((1 + o(1))q^{(d-h)(h-k)}\).

We are now ready to bound the third eigenvalue of \( M \) in the following lemma.

**Lemma 3.4.** The third eigenvalue of \( M \) is bounded by \(\sqrt{c_k(1 + o(1))q^{(d-h)h+k(2h-d-k+1)/2}}\), where \( c_k = (2k + 1)\left(\frac{k}{k/2}\right) \).

**Proof.** Let \( M \) be the adjacency matrix of \( G_P \), which has the form
\[
M = \begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix},
\]
where \( N \) is a \(|A| \times |B|\) matrix, and \( N_{KH} = 1 \) if \( K \in H \), and zero otherwise. Therefore,
\[
M^2 = \begin{bmatrix} NN^T & 0 \\ 0 & N^TN \end{bmatrix}.
\]
It follows from Lemma 2.2 that it suffices to bound the second eigenvalue of \( NN^T \). Given any two \( k \)-planes \( K_1 = \text{span}\{u_1, \ldots, u_k\} + u_{k+1} \) and \( K_2 = \text{span}\{v_1, \ldots, v_k\} + v_{k+1} \), we now count the number of their common neighbors, i.e. the number of \( h \)-planes containing both of them. We assume that \( H = \text{span}\{t_1, \ldots, t_h\} + t_{h+1} \) is a \( h \)-plane supporting \( K_1 \) and \( K_2 \). Then it follows from Lemma 3.1 and Lemma 3.2 that \( H \) can be written as \( H = \text{span}\{t_1, \ldots, t_h\} + u_{k+1} \) and \( u_1, \ldots, u_k, v_1, \ldots, v_k, u_{k+1} - v_{k+1} \in \text{span}\{t_1, \ldots, t_h\}. \) Thus the number of \( h \)-planes supporting \( K_1 \) and \( K_2 \) depends on the rank of the following system of vectors \( \{u_1, \ldots, u_k, v_1, \ldots, v_k, u_{k+1} - v_{k+1}\} \) which is denoted by \( \text{rank}(K_1, K_2) \). We also note that \( k + 1 \leq \text{rank}(K_1, K_2) \leq 2k + 1 \) since \( K_1 \) and \( K_2 \) are distinct. We assume that \( \text{rank}(K_1, K_2) = t \) and \( \text{span}\{u_1, \ldots, u_k, v_1, \ldots, v_k, u_{k+1} - v_{k+1}\} = \text{span}\{w_1, \ldots, w_t\} \) with \( w_i \in \mathbb{F}_q^d \) for \( 1 \leq i \leq t \), then the number of \( h \)-planes containing both \( K_1 \) and \( K_2 \) equals the number of \((h - t)\)-tuples of vectors \( \{x_1, \ldots, x_{h-t}\} \) in \( \mathbb{F}_q^d \) such that \( \text{rank}\{w_1, \ldots, w_t, x_1, \ldots, x_{h-t}\} = h \). Thus, the number of common neighbors of \( K_1 \) and \( K_2 \) is
\[
\frac{(q^d - q^1)(q^d - q^{t+1}) \cdots (q^d - q^{h-1})}{(q^h - q^1) \cdots (q^h - q^{h-1})} = (1 + o(1))q^{(d-h)(h-t)}.
\]
Therefore, \( NN^T \) can be presented as
\[
NN^T = q^{(d-h)(h-2k-1)}J + \left(q^{(d-h)(h-k)} - q^{(d-h)(h-2k-1)}\right)I \\
+ \sum_{k+1 \leq t \leq 2k} \left(q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}\right)E_t
\]
where $I$ is the identity matrix and $J$ is the all-one matrix, and for each $t$, $E_t$ are the adjacency matrix of the graphs $G(E_t)$: $V(G(E_t))$ is the set of all $k$-planes, and there is an edge between two $k$-planes $K_1 = \text{span}\{v_1, \ldots, v_k\} + u_{k+1}$ and $K_2 = \text{span}\{v_1, \ldots, v_k\} + v_{k+1}$ if and only if $\text{rank}(K_1, K_2) = t$. Note that these graphs are regular, and the degree of each vertex in each graph is counted as follows. For each $k + 1 \leq t \leq 2k$, and each vertex $K_1$, we now count the number of $k$-planes $K_2$ such that $\text{rank}(K_1, K_2) = t$. In order to do that, we consider two following cases:

1. If $u_{k+1} - v_{k+1} \in \text{span}\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$, then the number of $K_2$ is

$$
k!q^t(q^d - q^k) \cdots (q^d - q^{t-1})(q^t - q^{t-k}) \cdots (q^t - q^{t-k-1}) = (1 + o(1)) \left( \frac{k}{t-k} \right) q^{(t-k)(d-t+k+1)},
$$

where

- the term $(q^d - q^k) \cdots (q^d - q^{t-1})/(t - k)!$ is the number of $(t - k)$-tuples $v_1, \ldots, v_{t-k}$ such that rank$\{u_1, \ldots, u_k, v_1, \ldots, v_{t-k}\} = t$,
- the term $(q^t - q^{t-k}) \cdots (q^t - q^{t-k-1})/(2k-t)!$ is the number of $(2k - t)$-tuples of vectors $v_{t-k+1}, \ldots, v_k$ in span$\{u_1, \ldots, u_k, v_1, \ldots, v_{t-k}\}$ such that rank$\{v_1, \ldots, v_k\} = k$,
- the term $q^t$ is the number of choices of $u_{k+1} - v_{k+1}$ in span$\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$,
- the term $q^k(q^k - 1) \cdots (q^t - q^{t-k-1})/k!$ is the number of different ways to present a $k$-plane, and for each choice of $u_{k+1} - v_{k+1}$, then $v_{k+1}$ is determined uniquely.

2. If $u_{k+1} - v_{k+1} \notin \text{span}\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$, then the number of $K_2$ is

$$
k!(q^d - q^k) \cdots (q^d - q^{t-1})(q^{t-1} - q^{t-k-1}) \cdots (q^{t-1} - q^{t-k-1}) = o \left( q^{(t-k)(d-t+k+1)} \right),
$$

where

- the term $(q^d - q^k) \cdots (q^d - q^{t-1})/(t - k)!$ is the number of $(t - k)$-tuples $v_1, \ldots, v_{t-k-1}, u_{k+1} - v_{k+1}$ such that rank$\{u_1, \ldots, u_k, v_1, \ldots, v_{t-k-1}, u_{k+1} - v_{k+1}\} = t$,
- the term $(q^{t-1} - q^{t-k-1}) \cdots (q^{t-1} - q^{t-k-1})/(2k-t+1)!$ is the number of $(2k-t+1)$-tuples $v_{t-k}, \ldots, v_k$ in span$\{u_1, \ldots, u_k, v_1, \ldots, v_{t-k}\}$ such that rank$\{v_1, \ldots, v_k\} = k$,
- the term $q^k(q^k - 1) \cdots (q^t - q^{t-k-1})/k!$ is the number of different ways to present a $k$-plane.

Therefore, for each $t$, the degree of any vertex in $V(G(E_t))$ is

$$(1 + o(1)) \left( \frac{k}{t-k} \right) q^{(t-k)(d-t+k+1)}.$$
Let \( v_3 = (v_1, \ldots, v_{|A|}, u_1, \ldots, u_{|B|}) \) be the third eigenvector of \( M^2 \). Lemma 2.2 implies that \((v_1, \ldots, v_{|A|})\) is an eigenvector of \( NN^T \) corresponding to the eigenvalue \( \lambda_3^2 \). It follows from the equation (3.1) that

\[
(\lambda_3^2 - (q^{(d-h)(h-k)} - q^{(d-h)(h-2k-1)}))v_3 = \left( \sum_{k+1 \leq t \leq 2k} (q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}) E_t \right) v_3.
\]

Hence, \( v_3 \) is an eigenvector of

\[
\sum_{k+1 \leq t \leq 2k} (q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}) E_t.
\]

Since eigenvalues of sum of matrices are bounded by sum of largest eigenvalues of the summands, we have

\[
\lambda_3^2 \leq q^{(d-h)(h-k)} + q^{(d-h)(h-2k-1)} + k \binom{k}{t-k} (1 + o(1)) q^{(d-h)h+k(2h-d-k+1)}
\]

(3.2)

\[
\leq c_k (1 + o(1)) q^{(d-h)h+k(2h-d-k+1)},
\]

(3.3)

where \( c_k = (2k+1) \binom{k}{k/2} \). This completes the proof of the lemma.

\[\square\]

4 Proofs of Theorems 1.6, 1.9, and 1.10

Proof of Theorem 1.6. Since the degree of each \( k \)-plane is \((1 + o(1))q^{(d-h)(h-k)}\), and the number of \( h \)-planes is \((1 + o(1))q^{(d-h)(h+1)}\), we have

\[
\frac{\deg(A)}{|B|} = (1 + o(1)) q^{-(d-h)(k+1)}.
\]

Thus, Theorem 1.6 follows immediately from Lemma 2.1 and Lemma 3.4.

\[\square\]

Proof of Theorem 1.9. Theorem 1.9 follows immediately from Theorem 2.4 and Lemma 3.4.

\[\square\]

Proof of Theorem 1.10. Theorem 1.10 follows immediately from Theorem 2.3 and Lemma 3.4.

\[\square\]

5 Proof of Theorem 1.7

We first need the following definition.

Definition 5.1. We say that a set of \( k + 1 \) points in \( \mathbb{F}_q^d \) is in general position if no \((n - 1)\)-plane contains \((n + 1)\) of these points for all \( n \leq k \).

In order to prove Theorem 1.7, we need the following lemma.
Lemma 5.2. Any set $P'$ of $q^{k-1} + 1$ points in a $k$-plane $K \subset \mathbb{F}_q^d$ contains $k + 1$ points in general position.

Proof. The proof proceeds by induction on $k$. The base cases $k = 2$ and $k = 3$ are trivial. Suppose that the statement holds for $k - 1$, we now show that it also holds for $k$. Indeed, since $|P'| \geq q^{k-1} + 1$, there exists a hyperplane $H$ in $\mathbb{F}_q^d$ such that $|P' \cap H| \geq q^{k-2} + 1$. Therefore, it follows from induction hypothesis that $H \cap P'$ contains $k$ points $x_1, \ldots, x_k$ that are in general position. Note that if $K \subset H$, then we can repeat the process by replacing $\mathbb{F}_q^d$ by $H$. Thus, we can assume that $K \not\subset H$. This implies that $|K \cap H| \leq q^{k-1}$. Hence, there exists a point $x_{k+1} \in P'$ such that the set $\{x_1, \ldots, x_{k+1}\}$ is in general position. This completes the proof of the lemma. 

We are now ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7. Without loss of generality, we assume that $|P| = 3q^{d-1}$. Let $K_1$ be a set of all $k$-planes in $\mathbb{F}_q^d$ containing at most $q^{k-1}$ points from $P$. Then we have $I(P, K_1) \leq q^{k-1}|K_1|$. On the other hand, Theorem 1.6 implies that

$$I(P, K_1) \geq \frac{|P||K_1|}{q^{d-k}} - (1 + o(1))q^{\frac{k(d-k)}{2}}\sqrt{|P||K_1|}.$$

This implies that

$$\frac{|P||K_1|}{q^{d-k}} - (1 + o(1))q^{\frac{k(d-k)}{2}}\sqrt{|P||K_1|} \leq q^{k-1}|K_1|$$

Substituting $|P| = 3q^{d-1}$ gives us

$$|K_1| \leq \frac{3}{4}(1 + o(1))q^{k(d-k)+(d-1)-2k+2}.$$

Since the total number of $k$-planes in $\mathbb{F}_q^d$ is $(1 + o(1))q^{(d-k)(k+1)}$, the number of $k$-planes contains at least $q^{k-1} + 1$ points from $P$ is at least $(1 - o(1))q^{(d-k)(k+1)}$.

It follows from Lemma 5.2 that any $k$-plane $K$ satisfying $|K \cap P| \geq q^{k-1} + 1$ contains $k + 1$ points in general position from $P$, therefore, $K$ is spanned by $P$.

In short, the number of distinct $k$-planes determined by $P$ is $q^{d-1}$ when $|P| = 3q^{d-1}$, and the theorem follows. 

References


