Paths in pseudorandom graphs and applications

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Abstract
Let $G = (V,E)$ be an $(n,d,\lambda)$-graph. In this paper, we give an asymptotically tight condition on the size of $U \subset V$ such that the number of paths of length $k$ in $U$ is close to the expected number for arbitrary integer $k \geq 1$. More precisely, we will show that if $\lambda(n^2d) = o(|U|)$, then the number of paths of length $k$ in $U$ is $(1 + o(1))|U|^k |\frac{d}{n}|^k$. As applications, we obtain improvements and generalizations of recent results due to Bennett, Chapman, Covert, Hart, Iosevich, Pakianathan (2016).

1 Introduction
Let $G = G(n,p)$ be a random graph. For a fixed graph $H$ with $s \leq n$ vertices, $r$ edges, and automorphism group $\text{Aut}(H)$, it is well-known that the number of induced copies of $H$ in $G$ is

$$(1 + o(1))p^r (1 - p)^{\binom{s}{2} - r} \frac{n^s}{|\text{Aut}(H)|}.$$  

For a graph $G$, let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. The quantity $\lambda(G) = \max\{\lambda_2, -\lambda_n\}$ is called the second eigenvalue of $G$. A graph $G = (V,E)$ is called an $(n,d,\lambda)$-graph if it is $d$-regular, has $n$ vertices, and the second eigenvalue of $G$ is at most $\lambda$. It is well known that if $\lambda$ is much smaller than the degree $d$, then $G$ has certain random-like properties. Noga Alon [11] proved that every large subset of vertices in $(n,d,\lambda)$-graphs contains the “correct” number of copies of any fixed graph.

Theorem 1.1 (Alon, Theorem 4.10 [11]). Let $H$ be a fixed graph with $r$ edges, $s$ vertices, and maximum degree $\Delta$, and let $G = (V,E)$ be an $(n,d,\lambda)$-graph where $d \leq 0.9n$. Let $m < n$ satisfy $\lambda(n/d)^\Delta = o(m)$. Then, for every subset $U \subset V$ of cardinality $m$, the number of (not necessarily induced) copies of $H$ in $U$ is

$$(1 + o(1)) \frac{|U|^s}{|\text{Aut}(H)|} \left(\frac{d}{n}\right)^r.$$  

Note that if we take the ordering of vertex set into account in Theorem 1.1 then the number of copies of $H$ in $U$ is $(1 + o(1))|U|^s (d/n)^r$. In the case $H$ is a complete bipartite graph $K_{s,t}$, it has been shown by the second listed author [15] that the conditions on $d$ and $\lambda$ in Theorem 1.1 can be improved. Before presenting that result, we first need the following notations. Let $G \times G$ be the bipartite graph with two vertex parts $V(G)$ and $V(G)$. Two vertices $u$ and $v$ in different parts are connected by an edge if they are adjacent in $G$. For any two subsets $U_1, U_2 \subset V(G)$, let $G[U_1, U_2]$ be the induced bipartite subgraph of $G \times G$ on $U_1 \times U_2$.  

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Theorem 1.2 (Theorem 2.2, [15]). For any \( t \geq s \) and \( t \geq 2 \), let \( G = (V, E) \) be an \((n, d, \lambda)\)-graph. For every subsets \( U_1, U_2 \subseteq V \) with
\[
|U_1| |U_2| \geq \lambda^2 (n/d)^{t+s},
\]
the induced subgraph \( G[U_1, U_2] \) contains
\[
(1 + o(1)) \frac{|U_1|^s |U_2|^t}{s!t!} \left( \frac{d}{n} \right)^{st}
\]
copies of \( K_{s,t} \).

When either \( s \) or \( t \) is very small, one can further improve the bound in Theorem 1.2, for instance, in the case \( s = 2 \) and \( t \geq 1 \), the author of [15] indicated that under the condition \(|U_1||U_2| \geq \lambda^2 (n/d)^{t+1}\), the induced subgraph \( G[U_1, U_2]\) contains \((1 + o(1)) \frac{|U_1|^s |U_2|^t}{2!t!} \left( \frac{d}{n} \right)^{st}\) copies of \( K_{2,t} \).

Suppose \( U \) is set of vertices in an \((n, d, \lambda)\)-graph \( G \), and \( H \) is a path of length \( k \). It follows from Theorem 1.1 that if \( \lambda(n/d)^2 = o(|U|) \), then the number of copies of \( H \) in \( U \) is
\[
(1 + o(1)) |U|^{k+1} \left( \frac{d}{n} \right)^k.
\]
Our main purpose of this paper is to give an asymptotically tight condition on the size of \( U \subseteq V \) such that the number of paths of length \( k \) in \( U \) is close to the expected number for arbitrary \( k \geq 1 \). As applications, we obtain improvements and generalizations of results in [5]. Our first main result is as follows.

Theorem 1.3. Let \( G = (V, E) \) be an \((n, d, \lambda)\) graph. Suppose that \( U \subseteq V \) with \( \lambda \left( \frac{n}{d} \right) = o(|U|) \). For an integer \( k \geq 1 \), let \( P_k(U) \) be the number of paths of length \( k \) in \( U \), i.e.
\[
P_k(U) = \# \left\{ (u_1, \ldots, u_{k+1}) \in U^{k+1} : u_iu_{i+1} \in E(G), 1 \leq i \leq k \right\}.
\]
Then we have
\[
P_k(U) = (1 + o(1)) |U|^{k+1} \left( \frac{d}{n} \right)^k.
\]

On the sharpness of Theorem 1.3, we have the following.

Theorem 1.4. There exist an \((n, d, \lambda)\)-graph \( G \) and a set \( U \) of vertices with \(|U| = c\lambda \left( \frac{n}{d} \right)\) for some \( 0 < c < 1 \) such that \( P_k(U) = 0 \) for arbitrary \( k \geq 1 \).

We say that a path \((u_1, \ldots, u_{k+1}) \in U^{k+1}\) of length \( k \) is non-overlapping if \( u_i \neq u_j \) for all \( i \neq j \). For a set \( U \) of vertices in an \((n, d, \lambda)\)-graph \( G \), let \( D_k(U) \) be the number of non-overlapping paths of length \( k \) in \( U \), i.e.
\[
D_k(U) = \# \left\{ (u_1, \ldots, u_{k+1}) \in U^{k+1} : u_iu_{i+1} \in E(G), 1 \leq i \leq k, u_i \neq u_j, \forall i \neq j \right\}.
\]
In the following theorem, we show that under similar conditions on the size of \( U \), the number of non-overlapping paths of length \( k \) in \( U \) is \((1 - o(1)) |U|^{k+1} \left( \frac{d}{n} \right)^k\).
**Theorem 1.5.** Let \( G = (V, E) \) be an \((n,d,\lambda)\) graph. Suppose that \( U \subseteq V \) with \( \lambda \left( \frac{n}{d} \right) = o(|U|) \) and \( k \left( \frac{n}{d} \right) = o(|U|) \), then we have

\[
D_k(U) = (1 - o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]

Note that our results could be stated in multi-color versions, which will be useful for our later applications. Suppose that a graph \( G \) is edge-colored by a set of finite colors. We say that \( G \) is an \((n,d,\lambda)\)-colored graph if the subgraph of \( G \) on each color is an \((n, (1 + o(1))d, \lambda)\)-graph. The following are multi-color versions of Theorem 1.3 and Theorem 1.5.

**Theorem 1.6.** Let \( G = (V, E) \) be an \((n,d,\lambda)\)-colored graph. For a sequence \( c = (c_1, \ldots, c_k) \) of \( k \) colors, and \( U \subseteq V \), we define

\[
P^c_k(U) = \# \left\{ (u_1, \ldots, u_{k+1}) \in U^{k+1} : \text{the edge } u_iu_{i+1} \text{ is colored by } c_i, 1 \leq i \leq k \right\}.
\]

If \( U \) satisfies \( \lambda \left( \frac{n}{d} \right) = o(|U|) \), then we have

\[
P^c_k(U) = (1 + o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]

**Theorem 1.7.** Let \( G = (V, E) \) be an \((n,d,\lambda)\)-colored graph. For a sequence \( c = (c_1, \ldots, c_k) \) of \( k \) colors, and \( U \subseteq V \), we define

\[
D^c_k(U) = \# \left\{ (u_1, \ldots, u_{k+1}) \in P^c_k(U) : u_i \neq u_j \forall i \neq j \right\}.
\]

If \( U \) satisfies \( \lambda \left( \frac{n}{d} \right) = o(|U|) \) and \( k \left( \frac{n}{d} \right) = o(|U|) \), then we have

\[
D^c_k(U) = (1 + o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]

The proofs of Theorems 1.6 and 1.7 are similar to those of Theorems 1.3 and 1.5, respectively. To simplify the notation, we will only present the proofs of single-color results. Note that going from single-color formulations (Theorems 1.3 and 1.5) to multi-color formulations (Theorems 1.6 and 1.7) is just a matter of inserting different letters in a couple of places.

**1.1 Applications**

Let \( \mathbb{F}_q \) be a finite field of order \( q \), where \( q \) is an odd prime power. We denote the set of units in \( \mathbb{F}_q \) by \( \mathbb{F}_q^* \). For any two points \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) in \( \mathbb{F}_q^d \), we define the distance between \( x \) and \( y \) by

\[
\|x - y\| = (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2.
\]

Although it is not a norm, the function \( \|x - y\| \) has properties similar to the Euclidean norm (for example, it is invariant under orthogonal matrices and translations). The Erdős-Falconer distance problem asks for the minimum exponent \( \alpha \) such that for any set \( E \subseteq \mathbb{F}_q^d \) with \( |E| \gg q^\alpha \), the number of distinct distances determined by \( E \) is at least \( cq \) for some positive constant \( c \). Here and throughout, \( X \gg Y \) means that there exists \( C > 0 \) such that \( X \geq CY \). Bourgain, Katz, and Tao [3] considered a similar problem on the number of distinct distances determined
by a set of points in $\mathbb{F}_q^2$. Iosevich and Rudnev [9] proved that for any $E \subseteq \mathbb{F}_q^d$, if $|E| \geq 2q^{d+1}$, then all distances are determined by $E$. The authors of [7] indicated that the exponent $(d+1)/2$ is the best possible in odd dimensions. We refer the reader to [7] for more discussions.

Let $E$ be a set in $\mathbb{F}_q^d$, $d \geq 2$, and $k \geq 1$ be an integer. Let $t = (t_1, \ldots, t_k) \in \mathbb{F}_q^k$ with $t_i \neq 0$, $1 \leq i \leq k$, we define

$$P_k^t(E) = |\{(p_1, \ldots, p_{k+1}) \in E \times \cdots \times E : ||p_i - p_{i+1}|| = t_i, 1 \leq i \leq k\}|$$

as the number of paths of length $k$ in $E$ with given distances $(t_1, \ldots, t_k) \in \mathbb{F}_q^k$. In the case $k = 1$, we have $P_1^t(E)$ is the number of pairs $(x, y) \in E^2$ of distance $t_1$. In a recent work, Bennett, Chapman, Covert, Hart, Iosevich and Pakianathan [5], using Fourier analytic techniques, studied the magnitude of $P_k^t(E)$ for arbitrary $k \geq 1$ as follows.

**Theorem 1.8 (Bennett et al., [5]).** For $E \subseteq \mathbb{F}_q^d$, $d \geq 2$ and an integer $k \geq 1$. Suppose that $\frac{2k}{m^2 q^{d+1}} = o(|E|)$ then we have

$$P_k^t(E) = (1 + o(1)) \frac{|E|^{k+1}}{q^k}.$$

As a consequence of Theorem 1.8, the authors of [5] indicated that under the same condition as in Theorem 1.8, there exist non-overlapping paths of length $k$ in $E$ with arbitrary $k \geq 1$. The precise statement is as follows.

**Theorem 1.9 (Bennett et al., [5]).** Let $E$ be a set in $\mathbb{F}_q^d$, $d \geq 2$, and $k \geq 1$ be an integer. Let $t = (t_1, \ldots, t_k)$ with $t_i \neq 0$; $1 \leq i \leq k$, we define

$$D_k^t(E) = |\{(p_1, \ldots, p_{k+1}) \in E \times \cdots \times E : ||p_i - p_{i+1}|| = t_i, 1 \leq i \leq k; p_i \neq p_j, \forall i \neq j\}|.$$

Suppose that $|E| \geq \frac{2k}{m^2 q^{d+1}}$ then we have $D_k^t(E) > 0$.

Note that in the case $k = 1$, Theorem 1.9 implies the main result in [9]. In this section, we will present some improvements and generalizations of Theorems 1.8 and 1.9.

**The finite Euclidean distance graphs:** For a non-degenerate quadratic form $Q$ on $\mathbb{F}_q^d$, and $a \in \mathbb{F}_q^*$, the finite Euclidean distance graph $E_q(d, Q, a)$ is defined as follows:

$$V(E_q(d, Q, a)) = \mathbb{F}_q^d, \quad E(E_q(d, Q, a)) = \{(x, y) \in V \times V : Q(x - y) = a\}$$

The $(n, d, \lambda)$-form of $E_q(d, Q, a)$ has been studied in [2, 12] as follows.

**Theorem 1.10 ([2, 12]).** Let $Q$ be a non-degenerate quadratic form on $\mathbb{F}_q^d$. For any $a \in \mathbb{F}_q \setminus \{0\}$, the graph $E_q(d, Q, a)$ is an

$$(q^d, (1 + o(1))q^{d-1}, 2q^{(d-1)/2})$$

graph.

Let $G$ be a graph with the vertex set $\mathbb{F}_q^d$, and the edge between two vertices $x$ and $y$ are colored by the color $a$ if and only if $Q(x - y) = a$. Theorem 1.10 implies that the graph $G$ is a $(q^d, (1 + o(1))q^{d-1}, 2q^{(d-1)/2})$-colored graph with $(q - 1)$ colors. Thus as consequences of Theorems 1.6 and 1.7, we are able to improve Theorems 1.8 and 1.9 as follows.
Theorem 1.11. Let $\mathcal{E}$ be a set in $\mathbb{F}_q^d, d \geq 2$, and $k \geq 1$ be an integer. Let $t = (t_1, \ldots, t_k)$ with $t_i \neq 0$, $1 \leq i \leq k$, we define

$$P_k^t(\mathcal{E}) = |\{(p_1, \ldots, p_{k+1}) \in \mathcal{E} \times \cdots \times \mathcal{E} : Q(p_i - p_{i+1}) = t_i, 1 \leq i \leq k\}|.$$ 

Suppose that $q^{\frac{d+1}{2}} = O(|\mathcal{E}|)$, then we have

$$P_k^t(\mathcal{E}) = (1 + O(1))\frac{|\mathcal{E}|^{k+1}}{q^k}.$$ 

Theorem 1.12. Let $\mathcal{E}$ be a set in $\mathbb{F}_q^d, d \geq 2$, and $k \geq 1$ be an integer. Let $t = (t_1, \ldots, t_k)$ with $t_i \neq 0$, $1 \leq i \leq k$, we define

$$D_k^t(\mathcal{E}) = |\{(p_1, \ldots, p_{k+1}) \in \mathcal{E} \times \cdots \times \mathcal{E} : Q(p_i - p_{i+1}) = t_i, 1 \leq i \leq k, p_i \neq p_j, \forall i \neq j\}|.$$ 

Suppose that $kq = O(|\mathcal{E}|)$ and $q^{\frac{d+1}{2}} = O(|\mathcal{E}|)$, then we have

$$D_k^t(\mathcal{E}) = (1 + O(1))\frac{|\mathcal{E}|^{k+1}}{q^k}.$$ 

The finite upper half-plane graphs: For a finite field $\mathbb{F}_q$, the upper half plane on the finite field $\mathbb{F}_q$, which is denoted by $H_q$, is defined as

$$H_q = \{z = x + y\sqrt{\sigma} : x, y \in \mathbb{F}_q \text{ and } y \neq 0\},$$

where $\sigma$ is a non-square in $\mathbb{F}_q$. For any two points $z = u + v\sqrt{\sigma}$ and $w = x + y\sqrt{\sigma}$ in $H_q$, the distance between two points is

$$d(z, w) = \frac{(u - x)^2 - \sigma(v - y)^2}{vy}.$$ 

This distance is not a metric in the sense of analysis, but it is $GL(2, \mathbb{F}_q)$-invariant: $d(gz, gw) = d(z, w)$ for all $g \in GL(2, \mathbb{F}_q)$ and all $z, w \in H_q$.

For $a \in \mathbb{F}_q \setminus \{0, 4\sigma\}$, the finite upper half-plane graph $P(\sigma, a)$ is defined as follows: $V(P(\sigma, a)) = H_q \cup \{(z, w) \in E(P(\sigma, a)) \text{ if } d(z, w) = a\}$. The $(n, d, \lambda)$-form of $P(\sigma, a)$ has been established by Terras in [14].

Theorem 1.13 ([14]). For $a \in \mathbb{F}_q \setminus \{0, 4\sigma\}$, the finite upper half-plane graph $P(\sigma, a)$ is

$$\left(q^2 - q, q + 1, 2q^{1/2}\right) \text{-graph.}$$

Similarly, let $G$ be a graph with the vertex set $H_q$, and the edge between two vertices $z$ and $w$ are colored by the color $a$ with $a \neq 0, 4\sigma$ if and only if $d(z, w) = a$. Theorem 1.13 implies that the graph $G$ is a $(q^2 - q, q + 1, 2q^{1/2})$-colored graph with $(q - 2)$ colors. Therefore, as a consequence of Theorem 1.6, we have the following result.

Theorem 1.14. Let $\mathcal{E}$ be a set in $H_q$, and $k \geq 1$ be an integer. Let $t = (t_1, \ldots, t_k)$ with $t_i \neq 0$, $1 \leq i \leq k$, we define

$$P_k^t(\mathcal{E}) = |\{(p_1, \ldots, p_{k+1}) \in \mathcal{E} \times \cdots \times \mathcal{E} : d(p_i, p_{i+1}) = t_i, 1 \leq i \leq k\}|.$$ 

Suppose that $q^\frac{d}{2} = o(|\mathcal{E}|)$, then we have

$$P_k^t(\mathcal{E}) = (1 + O(1))\frac{|\mathcal{E}|^{k+1}}{q^k}.$$
2 Proofs of Theorems 1.3–1.5

To prove Theorems 1.3 and 1.5, we will need to use the following lemma.

**Lemma 2.1** (Chapter 9, [1]). Let \( G = (V, E) \) be an \((n, d, \lambda)\)-graph. For any two sets \( B, C \subseteq V \), the number of edges between \( B \) and \( C \) in \( G \), which is denoted by \( e(B, C) \), satisfies

\[
|e(B, C) - \frac{|B||C|}{n}| \leq \lambda \sqrt{|B||C|}.
\]

Suppose that \( B \) and \( C \) are two multi-sets of vertices in an \((n, d, \lambda)\)-graph. Let \( m_X(x) \) denote the multiplicity of \( x \) in \( X \), and \( e_m(B, C) \) be the number of edges with multiplicity between \( B \) and \( C \) in \( G \), by multiplicity we mean that if there is an edge between \( b \in B \) and \( c \in C \), then this edge will be counted \( m_B(b) \cdot m_C(c) \) times in \( e_m(B, C) \). For a multi-set \( X \), we still use the notation \(|X|\) for the cardinality of \( X \) which is the sum \( \sum_{x \in X} m_X(x) \). Recently, Hanson et al. [8] gave the following estimate on \( e_m(B, C) \) in an \((n, d, \lambda)\)-graph.

**Lemma 2.2** ([8]). Let \( G = (V, E) \) be an \((n, d, \lambda)\)-graph. The number of edges between two multi-sets of vertices \( B \) and \( C \) in \( G \) satisfies:

\[
|e_m(B, C) - \frac{|B||C|}{n}| \leq \lambda \sqrt{\sum_{b \in B} m_B(b)^2 \sqrt{\sum_{c \in C} m_C(c)^2}},
\]

where \( m_X(x) \) is the multiplicity of \( x \) in \( X \).

As a consequence of Lemma 2.2, we obtain the following recurrence relation between paths in \( U \).

**Lemma 2.3.** Let \( G \) be an \((n, d, \lambda)\)-graph. For a subset \( U \) of vertices, let \( P_k(U) \) be the number of paths of length \( k \) with vertices in \( U \). Then we have the following

\[
\bigg| P_{2k+1}(U) - \frac{dP_k(U)^2}{n} \bigg| \leq \lambda P_{2k}(U), \quad \bigg| P_{2k}(U) - \frac{dP_k(U)P_{k-1}(U)}{n} \bigg| \leq \lambda \sqrt{P_{2k}(U)P_{2k-2}(U)}.
\]

**Proof.** Let \( B \) and \( C \) be multi-sets defined as follows:

\[
B = \{v_{k+1}: (u_1, \ldots, u_{k+1}) \text{ is a path of length } k \text{ in } U\},
\]

\[
C = \{v_{k+2}: (u_{k+2}, \ldots, v_{k+2}) \text{ is a path of length } k \text{ in } U\}.
\]

One can check that \( P_{2k+1} \) is equal to the number of edges between \( B \) and \( C \) in the graph \( G \). Thus it follows from Lemma 2.2 that

\[
\bigg| P_{2k+1}(U) - \frac{dP_k(U)^2}{n} \bigg| \leq \lambda \sqrt{\sum_{b \in B} m_B(b)^2 \sqrt{\sum_{c \in C} m_C(c)^2}}.
\]

It is easy to see that \( \sum_{b \in B} m_B(b)^2 = \sum_{c \in C} m_C(c)^2 = P_{2k}(U) \). This implies that

\[
\bigg| P_{2k+1}(U) - \frac{dP_k(U)^2}{n} \bigg| \leq \lambda P_{2k}(U).
\]

By using the same arguments, we obtain

\[
\bigg| P_{2k}(U) - \frac{dP_k(U)P_{k-1}(U)}{n} \bigg| \leq \lambda \sqrt{P_{2k}(U)P_{2k-2}(U)},
\]

which completes the proof of the lemma. \( \Box \)
We will prove Theorem 1.3 by using induction on \( k \), so we need the following theorems for the base cases \( k = 1 \) and \( k = 2 \).

**Theorem 2.4.** Let \( G = (V, E) \) be an \((n, d, \lambda)\) graph. Suppose that \( U \subseteq V \) with \( \lambda \left( \frac{n}{d} \right) = o(|U|) \), then the number of paths of length one in \( U \) is \( (1 + o(1))|U|^2 \frac{d}{n} \).

**Proof.** The number of paths of length one is the number of edges between \( U \) and \( U \) in \( G \). Thus the theorem follows directly from Lemma 2.1.

**Theorem 2.5** (Theorem 3.3, [16]). Let \( G = (V, E) \) be an \((n, d, \lambda)\) graph. Suppose that \( U \subseteq V \) with \( \lambda \left( \frac{n}{d} \right) = o(|U|) \), then the number of paths of length two in \( U \) is \( (1 + o(1))|U|^3 \left( \frac{d}{n} \right)^2 \).

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3:** We first prove the upper bound of Theorem 1.3 by induction on \( k \). The base cases \( k = 1 \) and \( k = 2 \) follow from Theorems 2.4 and 2.5. Suppose that the statement holds for all \( 2k \geq 1 \). We now show that it also holds for \( 2k + 1 \) and \( 2k + 2 \). Indeed, it follows from Lemma 2.3 and induction hypothesis that

\[
P_{2k+1}(U) \leq \frac{d}{n} P_k(U)^2 + \lambda P_{2k}(U) \leq (1 + o(1)) \left( \frac{d}{n} \right)^{2k+1} |U|^{2k+2} + (1 + o(1)) \lambda \left( \frac{d}{n} \right)^{2k} |U|^{2k+1}
\]

when \( \lambda \left( \frac{n}{d} \right) = o(|U|) \).

For the case \( 2k + 2 \), it also follows from Lemma 2.3 that

\[
P_{2k+2}(U) \leq \frac{dP_k(U) P_{k+1}(U)}{n} + \lambda \sqrt{P_{2k}(U) P_{2k+2}(U)}.
\]

Solving this inequality in \( x = \sqrt{P_{2k+2}} \), we obtain

\[
P_{2k+2}(U) \leq \left( \lambda \sqrt{P_{2k}(U)} + \left( \frac{dP_k(U) P_{k+1}(U)}{n} \right)^{1/2} \right)^2.
\]

By using the induction hypothesis, we have

\[
P_{2k+2}(U) \leq (1 + o(1)) \left( \frac{d}{n} \right)^{2k+2} |U|^{2k+3}.
\]

In other words, we have proved that for all \( k \geq 1 \) and \( \lambda \left( \frac{n}{d} \right) = o(|U|) \)

\[
P_k(U) \leq (1 + o(1)) |U|^{k+1} \left( \frac{d}{n} \right)^k.
\]

By using the lower bounds of Lemma 2.3 and a nearly identical argument, we also obtain

\[
P_k(U) \geq (1 - o(1)) |U|^{k+1} \left( \frac{d}{n} \right)^k,
\]

under the condition \( \lambda \left( \frac{n}{d} \right) = o(|U|) \). This completes the proof of the theorem. \( \square \)
Proof of Theorem 1.4: Let \( d \geq 3 \) be an odd integer. From Theorem 1.10 we have that for any \( \lambda \in F_q^* \), the graph \( E_q(d, Q, \lambda) \) is an
\[
\left( q^d, (1 + o(1))q^{d-1}, 2q^{(d-1)/2} \right) \text{- graph.}
\]
Suppose \( Q(x) = x_1^2 + \cdots + x_q^2 \). It has been shown in [7, Theorem 2.7] that there exist a set \( U \subset F_q^d \) with \(|U| = c q^{(d+1)/2} = c \lambda n^{d/2} \) for some constant \( 0 < c < 1 \) and \( \beta \in F_q^* \) such that there are no two points in \( U \) of distance \( \beta \). This implies that there is no path of length \( k \) with arbitrary \( k > 1 \) in \( U \) in the graph \( E_q(d, Q, \beta) \). \( \square \).

In the proof of Theorem 1.5, we will use ideas given in [5, Corollary 1.3].

Proof of Theorem 1.5: Since the upper bound of Theorem 1.5 follows from Theorem 1.3, it suffices to prove that
\[
D_k(U) \geq (1 - o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]  
(2.1)
For \( u \in U \), let \( f_k(u) \) be the number of non-overlapping paths of length \( k \) in \( U \) beginning at \( u \in U \). Then we have
\[
D_k(U) = \sum_{u \in U} f_k(u).
\]
We now prove (2.1) by induction on \( k \). The base case \( k = 1 \) follows directly from Lemma 2.1. Suppose that the statement is true for all \( k - 1 \geq 1 \), we now show that it also holds for \( k \). Indeed, one can check easily that
\[
D_{k+1}(U) \geq \sum_{u \in U} f_k(u) (d_U(u) - k) = -kD_k(U) + \sum_{u \in U} f_k(u) d_U(u).
\]  
(2.2)
On the other hand, by using the same arguments as in the proof of Lemma 2.3, we have
\[
\left| \sum_{u \in U} f_k(u) d_U(u) - \frac{d D_k(U)|U|}{n}\right| \leq \lambda |U|^{1/2} \sqrt{\sum_{u \in U} f_k(u)^2} \leq \lambda |U|^{1/2} \sqrt{P_{2k}(U)},
\]
where we use the estimate \( \sum_{u \in U} f_k(u)^2 \leq P_{2k}(U) \). This implies that
\[
\sum_{u \in U} f_k(u) d_U(u) \geq \frac{d D_k(U)|U|}{n} - \lambda |U|^{1/2} \sqrt{P_{2k}(U)} \geq \frac{d D_k(U)|U|}{n} - \lambda (1 + o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]  
(2.3)
Putting (2.2) and (2.3) together gives us
\[
D_{k+1}(U) \geq \frac{D_k(U)|U|d}{n} - kD_k(U) - \lambda (1 + o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k.
\]
By using the induction hypothesis and the conditions \( \lambda \left( \frac{n}{d} \right) = o(|U|) \) and \( k \left( \frac{n}{d} \right) = o(|U|) \), we obtain
\[
D_{k+1}(U) \geq (1 - o(1))|U|^{k+1} \left( \frac{d}{n} \right)^k,
\]
which completes the proof of the theorem. \( \square \)
3 Concluding remarks

We conclude this paper with some remarks. Let $E_q(d, Q, a)$ be the finite Euclidean distance graph defined in the introduction. It follows from Theorem 1.1 that for $E \subset \mathbb{F}_q^d$, if $q^{\frac{d+3}{2}} = o(|E|)$ then $E$ contains many copies of a fixed triangle. Note that Theorem 1.1 can also be stated for $(n, d, \lambda)$-colored graphs, and in this form we have that the number of congruence classes of triangles in $E$ is $(1-o(1))q^3$ under the condition $q^{\frac{d+3}{2}} = o(|E|)$. However, this condition is only non-trivial when $d \geq 4$. If one can prove that under the same condition as in Theorem 1.3, i.e. $\lambda(n/d) = o(|E|)$, $E$ contains many copies of a fixed triangle, then this will imply that in the case $d = 2$, we only need the condition $q^{3/2} = o(|E|)$ to get almost all of congruence classes of triangles, which matches Iosevich’s conjecture [10] and the construction in [4]. Thus we are led to the following conjecture.

**Conjecture 3.1.** Let $G = (V, E)$ be an $(n, d, \lambda)$ graph. Suppose that $U \subseteq V$ with $\lambda \left(\frac{n}{d}\right) = o(|U|)$, then the number of copies of a fixed cycle $C$ of length 3 in $U$ is $(1 + o(1))|U|^3 (\frac{d}{n})^3$.

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References


