Lecture note 2

**Notation**

If $G$ is a graph, $V(G)$ is the vertex set of $G$ and $E(G)$ is the edge set of $G$. If $S \subset V(G)$, then $\overline{S} = V(G) \setminus S$. If $S, T \subset V(G)$, $E(S, T)$ denotes the set of edges with one endpoint in $S$ and one endpoint in $T$. For simplicity, we write $E(S)$ for $E(S, S)$.

**Isoperimetric inequalities**

Let $G$ be a graph. The *vertex boundary* of a set $S \subset V(G)$ is

$$\partial_v(S) = \{v \in \overline{S} : vw \in E(G) \text{ for some } w \in S\}.$$ 

Also, the *edge boundary* of $S$ is the set

$$\partial_e(S) = E(S, \overline{S}).$$

The isoperimetric inequality is concerned with the following question: Given a graph $G$, which set of given size minimizes the vertex/edge boundary of that set? Here, we consider this question for the graph of the hypercube $2[n]$, in which two vertices are connected if they differ in exactly one element. To be able to describe the set minimizing these boundaries, we introduce the following orderings on the hypercube. Let $A, B \in 2[n]$.

1. Simplicial ordering: $A <_S B$ if $|A| < |B|$, or $|A| = |B|$ and $\min(A \Delta B) \in A$.
2. Binary ordering: $A <_B B$ if $\max(A \Delta B) \in B$.

Also, for $k = 1, \ldots, 2^n$, let us denote by $I_{simp}(k), I_{bin}(k)$ the first $k$ elements of the Simplicial and Binary ordering, respectively. It turns out that initial segments of the Binary ordering minimize the edge-boundary.

**Theorem 1.** *(Edge-isoperimetric inequality)* Let $S \subset 2[n]$. Then

$$|\partial_e(S)| \geq |\partial_e(I_{bin}(|S|))|.$$ 

The idea of the proof is to perform compressions on $S$ that decrease the edge boundary and make $S$ resemble more and more to an initial segment of Binary.
Proof. We prove the statement by induction on $n$. If $n = 1$, the statement is trivial. Suppose that $n > 1$.

We define the $i$-compression of a family $S$ as follows. First, let

$$S_i^- = \{ A \in S : i \not\in A \}$$

and

$$S_i^+ = \{ A - i : i \in A \in S \}$$

denote the $i$-slices of $S$. Then the $i$-compression of $S$, denoted by $C_i(S)$, is the family $T$ that satisfies that $T_i^- = \mathcal{I}_{bin}(|S_i^-|)$ and $T_i^+ = \mathcal{I}_{bin}(|S_i^+|)$. (Here, we view $S_i^-$ and $S_i^+$ as elements of $2^{[n-1]}$ instead of elements of $2^{[n]-\{i\}}$, the elements of $[n] - \{i\}$ relabeled with the elements of $[n-1]$, keeping their order, and define the initial segments of binary accordingly.)

Claim 2. $|\partial_e(C_i(S))| \leq |\partial_e(S)|$.

Proof. Let $T = C_i(S)$. Note that

$$|\partial_e(S)| = |\partial_e(S_i^-)| + |\partial_e(S_i^+)| + |S_i^- \Delta S_i^+|.$$  

We can write $|\partial_e(T)|$ similarly.

By our induction hypothesis, we have $|\partial_e(T_i^-)| \leq |\partial_e(S_i^-)|$ and $|\partial_e(T_i^+)| \leq |\partial_e(S_i^+)|$. Also, as the initial segments of any ordering form a nested system, we have $T_i^- \subset T_i^+$ or $T_i^+ \subset T_i^-$. Either case, we have $|T_i^- \Delta T_i^+| \leq |S_i^- \Delta S_i^+|$. Hence, $|\partial_e(T)| \leq |\partial_e(S)|$. \hfill \qed

Say that a family $S$ is $i$-compressed, if $C_i(S) = S$. Also, a family $S$ is compressed if it is $i$-compressed for every $i = 1, ..., n$. By performing compressions on our family $S$, we arrive to a compressed family $T$. Indeed, if $S$ is not $i$-compressed, then $C_i(S)$ decreases the sum of the positions of its elements in the Binary order, so after a finite number of compressions, we must get a compressed family. By the previous claim, we have $|\partial_e(T)| \leq |\partial_e(S)|$. So we are done if we prove that any compressed family is an initial segment of Binary.

Unfortunately, this is not true. But luckily, it is almost true.

Claim 3. Let $T$ be a compressed family in $2^{[n]}$. Then $T$ is either an initial segment of Binary, or $T = (2^{[n-1]} - [n-1]) \cup \{\{n\}\}$.

Proof. Suppose that $T$ is not an initial segment of Binary. Then there exists $A <_B B$ such that $A \not\in T$ and $B \in T$. If there exists $i \in A \cap B$, then $A - i, B - i \in T_i^+$, which is impossible as $T_i^+$ is an initial segment of Binary in $2^{[n-1]}$. Similarly, if $i \not\in A$ and $i \not\in B$, then $A, B \in T_i^-$, which is also impossible. Hence, we must have that $A = [n] - B$.

But this is true for every $A <_B B$, $A \not\in T$ and $B \in T$. Therefore, $B$ must be the last element of $T$ in the Binary order, and $A$ must be the predecessor of $B$. This can only happen if $B = \{n\}$ and $A = [n-1]$. \hfill \qed

If $T = (2^{[n-1]} - [n-1]) \cup \{\{n\}\}$, then $|\partial_e(T)| = 2^{n-1} + 2n - 4$, while $\mathcal{I}_{bin}(|T|) = 2^{[n-1]}$ and $|\partial_e(\mathcal{I}_{bin}(|T|))| = 2^{n-1}$. Hence, we have $|\partial_e(T)| \geq |\partial_e(\mathcal{I}_{bin}(|T|))|$ in this case as well, finishing our proof. \hfill \qed
While Theorem 1 characterizes the sets minimizing the edge boundary, the minimum size of the edge boundary cannot be easily extracted from the theorem. To this purpose, we have the following quantitative bound. We omit its proof.

**Theorem 4.** Let $S \subseteq 2^{[n]}$. Then $|\partial_e(S)| \geq |S|(n - \log_2 |S|)$.

In case $|S| = 2^k$, this inequality is best possible as witnessed by the $k$-dimensional subcubes of $2^{[n]}$.

Now we show that initial segments of Simplicial minimize the vertex boundary. This is also known as Harper’s theorem.

**Theorem 5.** *(Vertex-isoperimetric inequality)* Let $S \subseteq 2^{[n]}$. Then

$$|\partial_v(S)| \geq |\partial_v(I_{\text{simp}}(|S|))|.$$

Similarly as in the proof of the edge-isoperimetric inequality, we apply compressions. The first half of the proof is more or less the same.

**Proof.** Let $B(S) = S \cup \partial_v(S)$. As $|B(S)| = |S| + |\partial_v(S)|$, it is enough to show that $|B(S)| \geq |B(I_{\text{simp}}(|S|))|$. We prove this statement by induction on $n$. If $n = 1$, the statement is trivial. Suppose that $n > 1$.

Define $S_i^-$ and $S_i^+$ as before. The $i$-compression of a family $S$, denoted by $C_i(S)$, is the family $T$ that satisfies that $T_i^- = I_{\text{simp}}(|S_i^-|)$ and $T_i^+ = I_{\text{simp}}(|S_i^+|)$.

**Claim 6.** $|B(C_i(S))| \leq |B(S)|$.

**Proof.** Let $T = C_i(S)$. Note that

$$|B(S)| = |B(S_i^-) \cup S_i^+| + |B(S_i^+) \cup S_i^-|.$$

We can write $|B(T)|$ similarly.

By our induction hypothesis, we have $|B(T_i^-)| \leq |B(S_i^-)|$ and $|B(T_i^+)| \leq |B(S_i^+)|$. Also, note that if $U$ is an initial segment of Simplicial, then $B(U)$ is also an initial segment of Simplicial. Hence, either $B(T_i^-) \subset T_i^+$ or $T_i^+ \subset B(T_i^-)$. In both cases, we have

$$|B(S_i^-) \cup S_i^+| \geq |B(T_i^-) \cup T_i^+|.$$

Similarly, we have $|B(S_i^+) \cup S_i^-| \geq |B(T_i^+) \cup T_i^-|$. Therefore, we get that $|B(S)| \geq |B(T)|$. \hfill $\Box$

Again, say that a family $S$ is $i$-compressed, if $C_i(S) = S$. Also, a family $S$ is compressed if it is $i$-compressed for every $i = 1, \ldots, n$. By performing compressions on our family $S$, we arrive to a compressed family $T$. Indeed, if $S$ is not $i$-compressed, then $C_i(S)$ decreases the sum of the positions of its elements in the Simplicial order, so after a finite number of compressions, we must get a compressed family. By the previous claim, we have $|B(T)| \leq |B(S)|$. We show that every compressed family is almost an initial segment of Simplicial.
Claim 7. Let $T$ be a compressed family in $2^{[n]}$. Then $T$ is either an initial segment of Simplicial, or $n = 2k$ and

$$T = \{ A \in 2^{[n]} : A <_{\text{simp}} \{1, k+2, k+3, \ldots, 2k\} \} \cup \{2, 3, \ldots, k+1\},$$
or $n = 2k + 1$ and

$$T = ([n]^{(\leq k)} - \{k+2, \ldots, 2k+1\}) \cup \{1, \ldots, k+1\}.$$

Proof. Suppose that $T$ is not an initial segment of Simplicial. Then there exists $A < B$ such that $A \not\in T$ and $B \in T$. If there exists $i \in A \cap B$, then $A - i, B - i \in T^+_i$, which is impossible as $T^+_i$ is an initial segment of Simplicial in $2^{[n-1]}$. Similarly, if $i \not\in A$ and $i \not\in B$, then $A, B \in T^-_i$, which is also impossible. Hence, we must have that $A = [n] - B$.

But this is true for every $A < B$, $A \not\in T$ and $B \in T$. Therefore, $B$ must be the last element of $T$ in the Simplicial order, and $A$ must be the predecessor of $B$. This can only happen when $T$ equals to one of the two exceptional sets described in the claim.

But if $T$ is one of the exceptional families described in the claim, then the vertex boundary of $T$ is larger than the vertex boundary of $I_{\text{simp}}(|T|)$, finishing or proof.

Corollary 8. Let $m = \sum_{i=0}^{k} \binom{n}{i}$ and let $S \subset 2^{[n]}$ of size $m$. Then $|\partial_v(S)| \geq \binom{n}{k+1}$.

Proof. We have $I_{\text{simp}}(m) = [n]^{(\leq k)}$ and $\partial_v(I_{\text{simp}}(m)) = [n]^{(k+1)}$. We are done by Theorem 5.

For $S \subset 2^{[n]}$ and $t \in \mathbb{Z}^+$, let $B_t(S)$ denote the $t$-neighborhood of $S$, that is,

$$B_t(S) = \{ A \in 2^{[n]} : |A \Delta B| \leq t \text{ for some } B \in S \}.$$

Corollary 9. Let $S \subset 2^{[n]}$ and let $t$ be a positive integer. Then $|B_t(S)| \geq |B_t(I_{\text{simp}}(|S|))|$.

Proof. Note that if $T$ is an initial segment of Simplicial, then so is $B_s(T)$ for every positive integer $s$. Also $B_{s+1}(S) = B_1(B_s(S))$, so the Corollary follows from repeated applications of Theorem 5.

Research problems

In this note, we considered the isoperimetric inequality for the hypercube. However, there are many natural graphs for which it is unknown what set minimizes the vertex/edge boundary. In many cases, it is conjectured that balls do minimize the vertex boundary, where a ball of radius $r$ and center $v \in V(G)$ in a graph $G$ is the set of all vertices whose distance from $v$ is at most $r$.

Conjecture 10. Let $n$ and $k$ be positive integers and let $G$ be the graph on $[n]^k$ in which $X$ and $Y$ are joined by an edge if $|X \Delta Y| = 2$. Then balls minimize the vertex boundary among sets of the same size.
Exercise problems

1. * Let $G$ be the usual graph on the $k \times k$ sized integer grid (that is, $(x, y)$ is connected to $(x+1, y), (x-1, y), (x, y+1)$ and $(x, y-1)$) and let $m \leq k^2$ be a positive integer. Which set of size $m$ minimizes the vertex boundary in $G$.

2. Let $S \subset 2^{[n]}$. Prove that $|E(S)| \leq |E(\text{I}_\text{bin}(|S|))|$. 

3. * A face of $2^{[n]}$ is a 4-cycle (equivalently, it consists of four sets $A, A \cup \{i\}, A \cup \{j\}$ and $A \cup \{i, j\}$, where $i, j \not\in A$ and $i \neq j$). Prove that among families $S \subset 2^{[n]}$ with $|S|$ given, initial segments of the Binary order maximize the number of 4-cycles in $S$. 
