Packing and covering
EPFL, 2014 Spring

12. Minkowski-Hlawka theorem

Remember that, given \( C \subset \mathbb{R}^d \) a convex, centrally symmetric body, we have denoted by \( \delta(C) \) - the maximal packing density of \( C \);
\( \vartheta(C) \) - the minimum covering density of \( C \).
Also remember that, during the exercise sessions, we have proved the following inequality
\[
\delta(C) \geq \frac{\vartheta(C)}{2^d} \geq \frac{1}{2d}.
\]

This provides a lower bound on the maximal packing density with respect to the minimum covering density. To prove this, we consider a saturated packing of \( C \) in the sense that we can add no more translates of \( C \), which is disjoint from the others. Let us denote this packing by \( C + C \). In this case, \( 2C + C \) forms a covering of \( \mathbb{R}^d \). This can easily be proven by contradiction: suppose there exist an \( x \) not covered by the copies of \( 2C \). That means, \( x \notin (2C + y) \).
On the other hand,
\[
(x + C) \cap (y + C) = \emptyset, \forall y \in C.
\]
From this, it follows that \( \exists c_1, c_2 \) such that \( x + c_1 = y + c_2 \) and thus \( x = y + (c_2 - c_1) \in 2C \) (remember that \( C \) is centrally symmetric). But this is a contradiction to the assumption above, which completes the proof.

**Goal:** We want now to extend this result to lattice packings. That means, we want to prove that there exist 'dense' lattice packings.

**Remark.** Let \( \Lambda \) be a lattice. Then, given that \( C \) is centrally symmetric, the following two propositions are equivalent:
(a) \( C + \Lambda \) is a packing;
(b) No lattice point apart from 0 is in \( 2C \).

**Definition.** A set \( C \subset \mathbb{R}^d \) is called star body if the following hold:
(i) \( 0 \in C \);
(ii) \( \forall x \in C, [0, x] \subseteq C \).

**Definition.** Let \( C \subseteq \mathbb{R}^d \) be a star body and \( \Lambda \subseteq \mathbb{R}^d \) a lattice. Then, we call \( \Lambda \) admissible for \( C \), if \( \Lambda \cap C = \{0\} \). We will define the critical determinant of \( C \) as
\[
\Delta(C) = \inf \{ \det \Lambda : \Lambda \text{ admissible for } C \}.
\]

**Goal:** We want to prove that \( \Delta(C) \) cannot be too large. We state the following theorem, without proof.

**Theorem 1** (Mahler). The following holds:
\[
\Delta(C) = \min \{ \det \Lambda : \Lambda \text{ admissible for } C \}.
\]
Thus, the minimum exists and it is attained for some lattice.
Minkowski’s theorem proved at the beginning of the course translates to the following formulation:

\[
\frac{\Delta(C)}{\text{Vol}(C)} \geq \frac{1}{2^d},
\]

which provides us with a lower bound for \(\frac{\Delta(C)}{\text{Vol}(C)}\).

In 1905, Minkowski proved the following upper bound for \(\frac{\Delta(C)}{\text{Vol}(C)}\) for the case \(C = B^d\):

\[
\frac{\Delta(C)}{\text{Vol}(C)} \leq \frac{1}{2\zeta(d)},
\]

where \(\zeta(d)\) is the Riemann zeta function, defined by

\[
\zeta(d) = 1 + \frac{1}{2^d} + \frac{1}{3^d} + \ldots
\]

This result was generalized to any centrally symmetric convex disc \(C\) by Hlawka in the 1940’s.

**Lemma 1.** Let \(f : \mathbb{R}^d \to \mathbb{R}\) be a continuous function vanishing outside a bounded region, and, for any real number \(\gamma\), set

\[
V(\gamma) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_{d-1}, \gamma) dx_1, \ldots dx_{d-1}.
\]

Furthermore, let \(\Lambda'\) be the integer lattice in the hyperplane \(x_d = 0\), and let \(\delta > 0\) be fixed. Given any vector \(y \in \mathbb{R}^d\) of the form \(y = (y_1, \ldots, y_{d-1}, \delta)\), let \(\Lambda_y\) denote the lattice in \(\mathbb{R}^d\) generated by \(\Lambda'\) and \(y\). Then

\[
\int_0^1 \cdots \int_0^1 \left( \sum_{x \in \Lambda_y, x \neq 0} f(x) \right) dy_1 \ldots dy_{d-1} = \sum_{i \in \mathbb{Z} \setminus \{0\}} V(i\delta).
\]

**Proof.** The proof of the theorem can be found in J.Pach, P.Agarwal, Combinatorial Geometry.

**Theorem 2** (Hlawka, 1944). Let \(g : \mathbb{R}^d \to \mathbb{R}\) be a bounded Riemann integrable function vanishing outside a bounded region, and let \(\epsilon > 0\). Then, there exists a unit lattice \(\Lambda\) in \(\mathbb{R}^d\), such that

\[
\sum_{0 \neq x \in \Lambda} g(x) < \int_{\mathbb{R}^d} g(x) dx + \epsilon.
\]

**Proof.** The proof of the theorem is based on the following can be found in J.Pach, P.Agarwal, Combinatorial Geometry.