Remember that last time we have proved the following lemma, which is a consequence of Blichfeldt’s lemma:

**Lemma 1.** Let \( \{B^d + c_i | i = 1, 2, \ldots \} \) be a packing and let \( D(c_i) \) be the Voronoi-Dirichlet cell corresponding to \( c_i \), \( D(c_i) = \{ x \in \mathbb{R}^d | \| x - c_i \| \leq \| x - c_j \|, \forall j \} \). Then, the distance from \( c_i \) to any \( d-k \)-dimensional face of \( D(c_i) \) is at least \( \sqrt{\frac{2k}{k+1}}, \forall k = 1, \ldots d \).

Note that \( B + c_i \) is always contained in \( D(c_i) \).

Let us now dissect \( D(c_1) \) into simplices of the form \( \text{conv}\{v_0, v_1, \ldots, v_d\} \) recursively, as follows:
- consider \( v_0 \) to be \( c_1 \) (note that \( c_1 \) lies inside the polytope \( D(c_1) \));
- divide \( D(c_1) \): for each face, take the convex hull of this face, together with \( c_1 \). This decompose \( D(c_1) \) as
\[
D(c_1) = \bigcup_{F_{d-1} \subseteq F_{d-2}} \text{conv}\{v_0, F_{d-1}\}.
\]
- the next vertex \( v_1 \) is chosen as the point of \( F_{d-1} \) the closest to \( v_0 \);
- once we have \( v_1 \), subdivide the face into simplices:
\[
D(c_1) = \bigcup_{F_{d-1} \subseteq F_{d-2}} \bigcup_{F_{d-2} \subseteq F_{d-1}} \text{conv}\{v_0, v_1, F_{d-2}\}.
\]
That means, at a given step, we have:
\[
D(c_1) = \bigcup \ldots \bigcup \text{conv}\{v_0, v_1, \ldots, v_{k-1}, F_{d-k}\}.
\]
In this case, we have one of the two cases: either \( d-k = 0 \) (and we stop), or it is not (and in this case we choose \( v_k \in F_{d-k} \) to be the closest to \( v_0 \).

Related to this decomposition, we can now state the following properties as a lemma:

**Lemma 2.** The Voronoi-Dirichlet cell \( D(c_1) \) can be decomposed into simplices of the form \( \text{conv}\{v_0, v_1, \ldots, v_d\} \),
where \( v_0 = c_1 \), and
(i) \( v_k \) lies in a \( (d-k) \)-dimensional face of \( D(c_1) \) containing \( v_k, v_{k+1}, \ldots, v_d \), and is the nearest point of this face to \( c_1 \) \((1 \leq k \leq d)\).
(ii) The scalar product
\[
\langle v_k - v_0, v_j - v_0 \rangle \geq \frac{2k}{k+1},
\]
for every \( 1 \leq k \leq j \leq d \).

**Proof.** The proof can be found in J.Pach, P.Aggarwal, Combinatorial Geometry. \qed

Using the above lemmas, one can now prove Rogers’ theorem (that gives the bound for sphere packing). The theorem is as follows:
Theorem 1 (Rogers). Let $S^d = \text{conv}\{p_0, ..., p_d\}$ be a regular simplex in $\mathbb{R}^d$, whose side length is 2. Draw a unit ball around each vertex of $S^d$. Let $\sigma_d$ denote the ratio of the volume of the portion of $S^d$, covered by balls, to the volume of the whole simplex, and let $\delta(B^d)$ be the density of the densest packing of unit balls in $\mathbb{R}^d$. Then
$$\delta(B^d) \leq \sigma_d.$$ 

Proof. The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

One can prove by some calculations that
$$\sigma_d = \left(\frac{1}{e} + o(1)\right) \cdot 2^{-d/2}$$
as $d \to \infty$, showing that for large values of $d$, Rogers’ bound is better than the bound of Blichfeldt by a factor of $2/e$. 