1. Which of the following statements are true for a probability space \((\Omega, P)\)?
   (a) If \(A, B\) are disjoint events (they cannot occur at the same time) then they are independent.
   (b) \(\Omega\) and \(A \subseteq \Omega\) are independent.
   (c) If \(A\) and \(B\) are independent then \(A\) and \(\overline{B}\) are also independent.

2. A random graph \(G(n, p)\) is a probability space, where for each pair \(1 \leq i < j \leq n\), \((i, j)\) is an edge of \(G(n, p)\) with probability \(p\), independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:
   (a) the expected number of edges in \(G(n, p)\)
   (b) the expected degree of a vertex in \(G(n, p)\)
   (c) the expected number of triangles (cycles of length 3) in \(G(n, p)\)
   (d) the probability that the degree of a given vertex \(v\) is at most \(k\)

3. Let \(X_1, \ldots, X_n\) be independent random variables such that for every \(i\), \(X_i = 1\) with probability \(p\) and \(x_i = 0\) otherwise. Let \(X = \sum_{i=1}^{n} X_i\). What is the expectation of \(X^2\)?

4. Let \(G\) be a graph with \(m\) edges, and let \(k\) be a positive integer. Prove that there is a coloring of the vertices of \(G\) with \(k\) colors so that at most \(m/k\) edges of \(G\) connect two vertices with the same color.

5. Let \(\mathcal{F}\) be a family of 3-element subsets of a set \(X\). Prove that the elements of \(X\) can be colored with 3 colors so that at least \(|\mathcal{F}| \cdot 3!/3^3\) sets in \(\mathcal{F}\) have exactly one element of each color.

6. In the lecture we gave a probabilistic argument that shows that any graph \(G\) on \(m\) edges contains a bipartite subgraph that has at least \(m/2\) edges. Show that the same argument actually proves the existence of a bipartite subgraph that has strictly more than \(m/2\) edges.

7. Consider the following algorithm on the graph \(G = (V, E)\). First split the vertices arbitrarily into two parts \(V_1\) and \(V_2\). Then in each step, if there is a vertex \(v \in V_i\) that has more neighbors in \(V_i\) than in \(V_{3-i}\), then move \(v\) to \(V_{3-i}\). Prove that this algorithm stops in a finite number of steps, and at the end the bipartite subgraph between \(V_1\) and \(V_2\) contains at least half of the edges of \(G\).

8*. After figuring out his last trick, you decide to pay your favorite magician another visit. This time he asks you to think of \(n\) arbitrary distinct reals \(a_1, \ldots, a_n\), and write down \(2^n - 1\) numbers: \(\sum_{i \in I} a_i\) for every nonempty subset \(I \subseteq \{1, \ldots, n\}\). He then figures out \(a_1, \ldots, a_n\). Prove that this is always possible if none of the sums you wrote down are equal to 0.