Geometric Graph Theory

9. Exercise, 28. April, 2010
Wednesday 1015-1145*, MA A1 10

1. Prove that if you have a sequence s of the numbers from 1 to n such that
(i) $s(i) \neq s(i+1)$ (where $s(i)$ denotes the i^{th} element of s),
(ii) if $i < j < k < l$ and $s(i) = s(k)$, $s(j) = s(l)$, then $s(i) = s(j)$,
then the length of s is at most $2n - 1$.

Definition: Denote by $\lambda_s(n)$ the length of the longest (n, s)-Davenport-Schinzel sequence, meaning a sequence containing n symbols such that no two consecutives are the same and it does not contain an alternation of a and b of length $s + 2$ as a subsequence for $a \neq b$. Thus the previous exercise claims that $\lambda_2(n) = 2n - 1$.

2. Prove that $\lambda_1(n) = n$.

3. Prove that in a maximal Davenport-Schinzel sequence the first and the last elements are the same if s is even.

4. Prove that any $(n, 2)$-Davenport-Schinzel sequence can be realized as the lower envelope of n (upwards) parabolas, meaning that for any sequence there is a set of n parabolas such that writing down which parabola is the smallest at each vertical line and deleting repetitions, we get the same sequence.
Example: $p_1(x) = (x + 1)^2, p_2(x) = x^2$, then the values of f are $2, 1$.

5. Show that the number of ways one can triangulate a convex $n + 1$-gon is the same as the number of $(n, 2)$-Davenport-Schinzel sequences of maximal length.
Hint: Number each edge of the triangulation with the smaller vertex, then read these numbers around each vertex and concatenate them.

6. (*HW*) Suppose we have n quadratic polynomials $p_1, \ldots, p_n$ from $\mathbb{R}$ to $\mathbb{R}$. For every x, consider the largest value of the $p_i(x)$'s and denote by $f(x)$ the value i where it is attained (if there is an equality, take the smallest i, so this sequence is similar to that of Exercise 4). At most how many times can $f(x)$ change?
Example: $p_1(x) = x, p_2(x) = x^2$, then the values of f are $2, 1, 2$.

7. * What is the smallest n for which $\lambda_3(n) \neq 3n - 2$?

New exercises and notes can be found at http://dcg.epfl.ch/page85509.html
Solutions to selected homeworks should be handed in at the beginning of the next session or sent to doemoetoer.palvoelgyi@epfl.ch.