Final Exam

Last name :                    First name :
Sciper :                      Section :

Exercise :  1  2  3  4  5  6  7  8  9  10  Σ
Grade :                      

- Write your answers in the space provided under each question.
- You may not use a calculator on this midterm.
- No additional materials are permitted.
- Even if you cannot solve a problem, write down your ideas.

Time : 8.15 – 11.15
Exercise 1:  (10 points: 2 each)
Write the definition for each of the following.

a. Independent set of vertices in a graph.
   Solution.

b. Chromatic number of a graph.
   Solution.

c. $k$-connected graph.
   Solution.

d. Ramsey number $R(r)$.
   Solution.

e. Line graph.
   Solution.
Exercise 2: (21 points: 3 each)

Answer the following questions. No argument is needed.

For Yes or No questions, 2 negative points will be given for a wrong answer.

1. Let $G$ be a connected graph with $n$ vertices and $T$ be a spanning tree of $G$. How many edges does $T$ have?
   
   Answer.

2. Let $G$ be a graph with $2n$ vertices and $M$ be a perfect matching of $G$. How many edges does $M$ have?
   
   Answer.

3. Is it true that every tree on 8 vertices has a perfect matching?
   
   Answer.

4. Let $G$ be a graph that contains a cycle. Is it true that $G$ has $K_3$ as a topological minor?
   
   Answer.

5. Does the graph of icosahedron (see Figure 1) have a $K_{3,3}$-minor?
   
   Answer.

6. Is it true that if a planar graph with $n$ vertices has $3n - 6$ edges then all its faces are triangular (contain exactly three vertices)?
   
   Answer.

7. What is the maximum number of edges that a graph on 7 vertices can have if it contains no $K_3$?
   
   Answer.
Exercise 3: (30 points: 5 each.)
For the following questions, answer them with Yes or No and justify your answers.

1. Does there exist a graph that is 3-edge connected but not 2-vertex connected?
Proof. Yes / No

2. Is the graph of a cube (shown below) planar?

![Cube Graph](image)

Proof. Yes / No

3. Can we draw $K_4$ on the plane in such a way that every vertex lies on the boundary of the outer face? If yes, draw it, if no, justify your answer.
Proof. Yes / No
4. Is it true that for every graph $G$ we have $\chi(G) \leq 1 + \text{average degree of } G$?
Proof. 
Yes / No

5. Is it true that the number of graphs on 6 vertices with an Eulerian cycle is at most the number of graphs on 6 vertices that do not have an Eulerian cycle?
Proof. 
Yes / No

6. Does it follow from 5 that the probability that a random graph $G(6, 1/2)$ with 6 vertices and edge probability 1/2 contains an Eulerian cycle is $\leq 1/2$?
Proof. 
Yes / No
Exercise 4: (20 points: 5 for each part.)

Let $G$ be the icosahedron graph (drawn in Figure 1).

Use the drawings of $G$ in Figure 1 to justify your answer.

a. Find a matching of maximum size in $G$.
b. Find $\alpha(G)$.
c. Find $\chi(G)$.
d. Find $\chi'(G)$. 
Figure 1: The icosahedron graph
Exercise 5: (10 points.)

For two graphs $H_1, H_2$, we defined $R(H_1, H_2)$ to be the least number such that for every graph $G$ with at least $R(H_1, H_2)$ vertices either $H_1$ is a subgraph of $G$ or $H_2$ is a subgraph of $\overline{G}$ (the complement of $G$). What is $R(K_{1,5}, K_{1,5})$?
Exercise 6: (10 points.)

Denote by $H$ the graph obtained from $K_4$ by deleting one edge. Calculate the expected number of induced copies of $H$ in $G(n, p)$. Here $G(n, p)$ is a random graph model with edge probabilities equal to $p$. 
Exercise 7: (15 points.)
If \( \binom{n}{k} \left( \frac{1}{2} \right)^{k^2-1} < 1 \) then there is a 2-coloring of the edges of \( K_{n,n} \) such that it does not contain a monochromatic copy of \( K_{k,k} \).

Hint: Argue as in the proof of the lower bound for \( R(r) \).
Exercise 8: (15 points.)
Show that for every graph \( G \) on \( n \) vertices we have \( \chi(G) + \chi(G) \leq n + 1 \).

**Hint:** Use induction on \( n \).
**Exercise 9:** (15 points.)

For any two graphs $H_1, H_2$, we defined $R(H_1, H_2)$ to be the least number such that for every graph $G$ with at least $R(H_1, H_2)$ vertices either $H_1$ is a subgraph of $G$ or $H_2$ is a subgraph of $\overline{G}$ (the complement of $G$).

Show that $R(K_s, K_t) \leq R(K_{s-1}, K_t) + R(K_s, K_{t-1})$.

**Hint:** Take a vertex and look at its neighbors in $G$ and in $\overline{G}$. 
Exercise 10: (15 points.)
Construct a 3-regular graph without a perfect matching.

Hint: Make use of Tutte’s theorem on matchings in graphs.