Let \( K_n \) denote a complete graph with \( n \) vertices. Given any positive integers \( k \) and \( l \), the **Ramsey number** \( R(k, l) \) is defined as the smallest integer \( n \) such that in any two-coloring of the edges of \( K_n \) by red and blue, either there is a red \( K_k \) or a blue \( K_l \).

1. (i) Prove that for every positive integer \( n \) we have
\[
R(k, k) > n - \binom{n}{k} 2^{1 - \binom{k}{2}}.
\]
(ii) Use part (i) to show that there exists a constant \( c \) such that for every \( k \geq 2 \) we have
\[
R(k, k) \geq (1 - \frac{c}{k}) \cdot \frac{k}{e} \cdot 2^{k/2}.
\]

2. (i) Prove that for every positive integer \( n \) and every real number \( p \) with \( 0 \leq p \leq 1 \), we have
\[
R(4, k) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{4} (1 - p)^6.
\]
(ii) Prove that there exists a constant \( c \) so that for every positive integer \( k \geq 2 \) we have
\[
R(4, k) \geq \frac{ck^2}{(\ln k)^2}.
\]

3. Consider the 3-uniform hypergraph \( H = (V, E) \) with the vertex set \( V \) and the set of hyperedges \( E \), that is, a system of 3-element subsets of \( V \). We call the set \( A \subset V \) an independent set for \( H \), if for every hyperedge \( e \in E \), we have \( e \not\subset A \). Prove that if \( |V| = n, |E| = m \geq n/3 \), then \( H \) contains an independent set of size at least \( \frac{2n^3/2}{3\sqrt{3m}} \).

4. For the random variable \( X \), we define the variance as
\[
\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2).
\]
Denote \( \mu = \mathbb{E}(X) \), \( \sigma^2 = \text{Var}(X) \). If \( \mu \) and \( \sigma^2 \) are finite with \( \sigma^2 \) being nonzero, then **Chebyshev’s inequality** states that for every positive real number \( r \) we have
\[
\Pr(|X - \mathbb{E}(X)| \geq r\sigma) \leq \frac{1}{r^2}.
\]
Use Chebyshev’s inequality to show that for every positive integer \( n \) we have
\[
\sum_{|k| < \sqrt{n}} \binom{2n}{n+k} \geq 2^{2n-1}.
\]

5. Consider the graph \( G = (V, E) \) with \( |V| = n > 2 \). Prove that there exist two vertices \( u, v \in V \) such that there are at least \( \frac{n-3}{2} \) vertices in \( V \setminus \{u, v\} \), which are adjacent to either both or neither of \( u, v \).

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\$ – an optional contest problem. You may submit a solution until the beginning of the next lecture.