1. Let $K_n$ denote a complete graph with $n$ vertices. Given any positive integers $k$ and $l$, the Ramsey number $R(k, l)$ is defined as the smallest integer $n$ such that in any two-coloring of the edges of $K_n$ by red and blue, either there is a red $K_k$ or a blue $K_l$.

(i) Prove that if there is a real $p$, $0 \leq p \leq 1$ such that

$$\left( \frac{n}{3} \right) p^3 + \binom{n}{k} (1-p)^{\binom{k}{2}} < 1,$$

then we have $R(3, k) > n$.

(ii) Use the previous part to show that there exists a constant $c$ such that

$$R(3, k) \geq \frac{ck}{\ln k}.$$

2. Suppose $n \geq 3$ and let $H$ be an $n$-uniform hypergraph, i.e. each edge of $H$ contains exactly $n$ vertices. Prove that if the number of edges of $H$ is at most $3n - \frac{1}{2} n^2$, then there exists a coloring of the vertices of $H$ by 3 colors so that in every edge all 3 colors are represented.

3. Suppose $n \geq 2$ is a natural number and denote the set of residues modulo $n^2$ by $\mathbb{Z}_{n^2} = \{0, 1, 2, \ldots, n^2 - 1\}$. Consider $A \subset \mathbb{Z}_{n^2}$ with $|A| = n$. Prove that there is a set $B \subset \mathbb{Z}_{n^2}$ of size $n$ such that at least half of the elements of $\mathbb{Z}_{n^2}$ can be written as $a + b$ modulo $n^2$, where $a \in A$ and $b \in B$.

4. Let $n \geq 2$ be a natural number and consider the set $A$ consisting of $n$ integers. Define $A + A$ as

$$A + A = \{a + a' : a \neq a', a, a' \in A\}.$$

Prove that $|A + A| \geq 2n - 3$, and this bound is the best possible.

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$\$ — an optional contest problem. You may submit a solution until the beginning of the next lecture.