1. Suppose that \( f : \mathbb{N} \rightarrow \mathbb{N} \) is a nondecreasing function satisfying the recursion \( f(3n) \geq 3f(n) + cn \) for some positive constant \( c \) and every \( n \in \mathbb{N} \). Prove that there is a positive constant \( c' \) such that \( f(n) > c'n \log n \).

2. Construct a geometric graph with \( n \) vertices, at least \( 4n - o(n) \) edges, and no three pairwise crossing edges. Can you find such a graph with at least \( 4n - c \) edges for some constant \( c \)?

3. Let \( G \) be a random graph with \( n \) vertices, where each edge of \( G \) is selected independently with probability \( 1/2 \). Prove that for sufficiently large \( n \), with probability at least \( 1/2 \), the bisection width of \( G \) satisfies \( b(G) \geq n^2/100 \).

4. Prove, using the crossing lemma, that if no four edges of a geometric graph \( G \) with \( n \) vertices are pairwise crossing, then \( |E(G)| < cn^{7/4} \) for a suitable constant \( c \).

5. For some constants \( c_1, c_2 > 0 \) and arbitrarily large \( n \), prove that there is a graph \( G_n \) with \( n \) vertices, at most \( c_1 n \) edges, crossing number at least \( c_2 n^2 \), and such that every subgraph of \( G_n \) with \( m \) vertices has at most \( c_1 m \) edges.

---

\( \ddot{\smile} \) — an optional homework. You may submit a written solution of this problem until the beginning of the next lecture and receive our feedback.

\( \ddagger \) — an optional contest problem. You may submit a solution until the beginning of the next lecture. We encourage you to participate.

New exercises and notes can be found at \url{http://dcg.epfl.ch/page-117408-en.html}. 