A binary relation $\preceq$ on a set $X$ is a partial order on $X$ if $\preceq$ is reflexive, antisymmetric and transitive. Two elements of $X$ are comparable by $\preceq$ if $x \preceq y$ or $y \preceq x$, otherwise they are incomparable. A partial order $\preceq$ on $X$ is a total order if every two elements of $X$ are comparable. The comparability graph $G(P)$ of a partially ordered set $P = (X, \preceq)$ is the graph with vertex set $X$ such that for every two distinct elements $x, y \in X$, $xy$ is an edge of $G(P)$ if and only if $x$ and $y$ are comparable by $\preceq$. A chain is a totally ordered subset. An antichain is a subset of elements that are pairwise incomparable.

1. Show that every partial order on a set $X$ with $n$ elements can be extended to a total order on $X$.

2. Show that $C_5$ is not a comparability graph.

3. Let $P_n = (\{1, 2, \ldots, n\}, |)$ be a partially ordered set where $k|m$ means that $m$ is divisible by $k$. Find
   a) a longest chain in $P_n$,
   b) a largest antichain in $P_n$.

4. Consider the following partial order on the set of vectors in $\mathbb{R}^d$ with positive integer coordinates: $(n_1, n_2, \ldots, n_d) \preceq_d (m_1, m_2, \ldots, m_d)$ if and only if for every $i = 1, 2, \ldots, d$, we have $n_i \leq m_i$. Prove that
   a) $(\{1, 2, \ldots\}^2, \preceq_2)$ has no infinite antichain,
   b) $(\{1, 2, \ldots\}^3, \preceq_3)$ has no infinite antichain.

5. Let $G$ be a complete graph drawn in the plane so that every two edges cross at most once and adjacent edges do not cross. For an edge $e$, we define a graph $H_e = (V(G), F_e)$, where $F_e$ is the set of edges of $G$ that cross $e$. Prove that $H_e$ is a comparability graph.