1. Let $X$ be a random variable taking non-negative integer values. Prove that

$$\Pr(X = 0) \leq \frac{\text{Var}(X)}{\mathbb{E}(X^2)}.$$ 

2. Consider a sequence $\{X_1, X_2, X_3, \ldots\}$ of non-negative random variables such that

$$\lim_{n \to \infty} \frac{\text{Var}(X_n)}{\mathbb{E}(X_n)^2} = 0.$$ 

Use Chebyshev’s inequality to show that

$$\lim_{n \to \infty} \Pr(X_n > 0) = 1.$$ 

3. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be $n$ two-dimensional vectors, where $x_i, y_i$ are integers with $|x_i|, |y_i| \leq \frac{2^n}{100\sqrt{n}}$, for each $i$ ($1 \leq i \leq n$). Prove that there exist two disjoint sets $I, J \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$ 

4. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exist numbers $a_1, a_2, \ldots, a_n$, each taking a value in $\{-1, 1\}$, such that

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \leq 2\left(\sum_{i=1}^{n} x_i^2\right).$$ 

(Hint: Use Chebyshev’s inequality!)

§ 5. There are several circles of total circumference 10 inside a square of side length 1. Prove that there is a line that intersects at least 4 of these circles.

§ — an optional contest problem. You may submit a solution until the beginning of the next lecture.