Theorem 1. Let $G = (V, E)$ be a graph on $n$ vertices, such that $G$ does not contain $K_3$ as subgraph and $|E| = ex(n, K_3) - t$ for some non-negative integer $t$. Then one can remove at most $t$ edges from $G$ to make it bipartite.

Proof. Let $v \in V$ be a vertex of maximum degree in $G$ and let us denote its degree by $\Delta$. We denote by $N(v)$ the set of neighbors of $v$ in $G$, and we define $B := N(v)$ and $A := V \setminus N(v)$. Since $v$ is of maximum degree, we have that $|B| = \Delta$ and $|A| = n - \Delta$.

For any subgraphs $H_1$ and $H_2$ of $G$, we denote by $E(H_1, H_2)$ the set of edges of $G$ with one extremity in $H_1$ and the other in $H_2$. We use the notation $E(H)$ for $E(H, H)$.

Let us first observe that there are no edges with both endpoints in $B$. Indeed, let us assume that there are two vertices $v_1, v_2 \in B$ such that $(v_1, v_2)$ is an edge in $G$. Then the vertices $v, v_1, v_2$ form a $K_3$.

We prove now that $|E(A)| \leq t$. If this is indeed the case, the deletion of the edges in $E(A)$ will provide us a bipartite graph with bipartition $A \cup B$ on the same set of vertices as $G$ (remember that there are no edges running between vertices of $B$, and after deletion there will be no edges running between vertices of $A$).

Since the cardinality of $A$ is $n - \Delta$, and the degree of each vertex $u \in A$ is at most $\Delta$, we have that

$$\Delta(n - \Delta) \geq \sum_{u \in A} d(u).$$

On the other hand, we have that

$$\sum_{u \in A} d(u) = 2|E(A)| + |E(A, B)| =$$

$$= |E(A)| + (|E(A) + |E(A, B)|) = |E(A)| + |E(G)|.$$

Since $|E(G)| = ex(n, 3) - t$, and the number of edges in $K_{\Delta, n-\Delta}$ (which is $\Delta(n - \Delta)$) cannot exceed $ex(n, K_3)$, we have that $|E(G)| \geq \Delta(n - \Delta) - t$.

This, together with the two inequalities above, imply that $|E(A)| \leq t$. We can now remove all the edges of $E(A)$ to obtain a bipartite graph, which completes the proof. \(\square\)