Questions

1. We will prove the following equality:

\[
\sum_{k=0}^{|n/2|} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n + 1
\]  

(1)

- Define \(\{0,1\}^n\) to be the set of all binary strings of length \(n\). For each \(i, 1 \leq i \leq n - 1\), define the following set:

\[A_i = \{(x_1, \ldots, x_n) \in \{0,1\}^n : x_i = 0, x_{i+1} = 1\}\]

\(A_i\) is the set of all binary strings of length \(n\) with 0 in the \(i\)-th position, and 1 in the \((i+1)\)-th position. Prove that

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = \binom{n-k}{k} \cdot 2^{n-2k}
\]

- Prove that \(|\{0,1\}^n - \bigcup_{1 \leq i \leq n-1} A_i| = n + 1\).
- Prove the identity given in equation (1).

**Solution:** i) Count the number of strings filling the \(k\) given places with 01’s (and the rest of the places with all possible combinations). And sum this over all possible \(k\) places. ii) Counting the complement. iii) Use inclusion-exclusion.

2. How many ways are there to seat \(n\) couples in a row of \(2n\) chairs such that the couples never sit next to each other?

**Solution:** Inclusion-Exclusion on the complement problem.

3. How many ways are there to distribute \(n\) identical chocolates to \(k\) (non-identical!) children such that no child gets more than \(m - 1\) chocolates?

**Solution:** Inclusion-Exclusion on the complement problem.

4. Prove the following upper bound: \(n! \leq e \sqrt{n(n/e)^n}\). Use the method of integration, carefully dealing with the triangle areas.

**Solution:** Exactly as done in the class, except one has to exclude the \(n\) triangle areas.

5. Prove Bernoulli’s Inequality: for any natural number \(n\) and real \(x \geq -1\): \((1 + x)^n \geq 1 + nx\).

**Solution:** Induction on \(n\).

6. Prove the following estimate by induction on \(k\): \(\binom{n}{k} \leq (en/k)^k\)

**Solution:** Induction in a very similar way to the one given in the Matousek-Nesetril book.

**Bonus Problem.** Let \(n\) be an even integer. Find the number of distinct strings of length \(n\) that can be obtained by concatenating copies of the strings 0, 10 and 11. For example, 0101110 is a valid string (0 10 11 10) but 1100101 is not.
Solution: By induction. Start from the right-most bit. If that is a 0, then any string on the earlier $n - 1$ bits together with this last bit 0 will form a valid codeword. So that is $2^{n-1}$ codewords ending with a 0. Now, if the last bit is a 1, then the second-last bit has to be a 1 also, and also has to form the 11 with the last bit (no other way for a valid codeword). This gives the recurrence $f(n) = 2^{n-1} + f(n - 2)$, which solves to $(2^{n+1} + 1)/3$. 

10 points.