You can hand in one of the following problems at the start of Tuesday’s problem session. Please explain your solution carefully. Don’t forget to put your name.

NP-Hardness

1. Show that the following two optimization problems are \( \mathcal{NP} \)-hard:

   **Independent Set:** Given an undirected graph \( G \), find a maximum cardinality independent set, i.e. a set \( I \subset V(G) \) such that \( E(I) = \{ uv \in E(G) : u, v \in I \} = \emptyset \).

   **Clique:** Given an undirected graph \( G \), find a maximum cardinality clique, i.e. a \( K \subset V(G) \) such that \( uv \in E(G) \) for all \( u, v \in K \).

2. Show that the following optimization problem is \( \mathcal{NP} \)-hard:

   **Longest Path:** Given a directed graph \( G \) with weights \( w : E(G) \to \mathbb{R} \), and \( s, t \in V(G) \), find a maximum weight directed path from \( s \) to \( t \).

3. Show that the following optimization problem is \( \mathcal{NP} \)-hard:

   **Integer Programming:** Given a matrix \( A \in \mathbb{Z}^{m \times n} \) and vectors \( b \in \mathbb{Z}^m, c \in \mathbb{Z}^m \), find a vector \( x \in \mathbb{Z}^n \) such that \( Ax \leq b \) and \( cx \) is maximum, if possible.

4. Show that the following optimization problem is \( \mathcal{NP} \)-hard:

   **Metric TSP:** Let \( G \) be a complete undirected graph \( G \) with a weight function \( d : E(G) \to \mathbb{R}_{>0} \) that satisfies the triangle inequality
   \[
   d(uw) \leq d(uv) + d(vw)
   \]
   for all \( u, v, w \in V(G) \).
   Find a minimum weight Hamilton cycle in \( G \).