Matroid Intersection

1. Show that the problem of finding a Hamilton path from s to t in a given directed graph D can be solved using an intersection of 3 matroids.

First remove all edges in $\delta^\text{in}(s)$ and $\delta^\text{out}(t)$ from D.

We take the following three matroids with $X = E(D)$:

\[
\mathcal{M}_1 = P(\{\delta^\text{in}(v)\}), \\
\mathcal{M}_2 = P(\{\delta^\text{out}(v)\}), \\
\mathcal{M}_3 = F(G).
\]

An edge set that is independent with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ has at each vertex at most one incoming edge and at most one outgoing edge. Hence it consists of cycles and paths. If it is also independent with respect to $\mathcal{M}_3$, then there are no cycles, so it is a disjoint union of paths.

We claim that D has a Hamilton path if and only if a maximum independent set in this intersection of 3 matroids has $|V(D)| - 1$ edges.

Let I be a maximum common independent set. If it is a spanning tree, then it must be one single path, which must then be a Hamilton path from s to t (because we removed those edges at the start).

On the other hand, if I is not spanning tree, then it has at most $|V(D)| - 2$ edges. That means there is no Hamilton cycle, because that would be a common independent set with $|V(D)| - 1$ edges.

2. Given an undirected graph $G = (V,E)$, an orientation is a directed graph $D = (V,E')$ with a bijection $\varphi : E' \to E$ such that $\varphi(ab) = \{a,b\}$. In other words, each edge $\{a,b\} \in E$ is given a direction, either $ab$ or $ba$.

Given $k : V \to \mathbb{N}$, show that the problem of finding an orientation such that

\[
\delta^\text{in}(v) = k(v)
\]

for each $v \in V$, or showing that none exists, can be solved using the matroid intersection algorithm.

We just have to give two matroids such that a common independent set is an orientation satisfying that condition. Let X be the set of directed edges $ab$ and $ba$ for each edge $\{a,b\} \in E(G)$. Then the matroids are

\[
\mathcal{M}_1 = P(\{uw,vu\}_{u,v} \in E(G)), \\
\mathcal{M}_2 = \{Y \subset X : |Y \cap \delta^\text{in}(v)| \leq k \forall v \in V(G)\}
\]

That the second one is a matroid is proved just like for partition matroids.

A common independent set would be an orientation because of the first matroid (one direction per edge), and it would satisfy $\delta^\text{in}(v) \leq k(v)$ for all $v$ due to the second matroid.

Then an orientation as required exists if and only if there is a common independent set of size $\sum_{v \in V(G)} k(v)$. 
3. Use the matroid intersection algorithm to show that there is no simultaneous spanning tree in the following two graphs (i.e., there is no \( T \subset \{a, b, \ldots, j\} \) that is a spanning tree in both).

If we do the greedy part of the algorithm in alphabetical order, then it would find the common independent set \( \{a, c, e, g\} \) (if we used a different order, we would get something “isomorphic”). It is not a spanning tree in either graph. If we now draw the directed graph as in the matroid intersection algorithm, we’ll see that there is no path from \( X_1 = \{f, h, i\} \) to \( X_2 = \{b, d, j\} \). Alternatively, we could observe that

\[
U = \{a, b, c, d, j\}
\]

(the set of vertices in \( D_I \) from which \( X_2 \) can be reached) gives equality in \(|I| \leq r_{M_1}(U) + r_{M_2}(X - U)\), which implies that \( I \) is maximal (see the matroid intersection theorem).

4. Make up 2 matroids such that the matroid intersection algorithm needs at least 2 non-greedy steps (i.e., with \(|Q| > 1\)) to get a maximum common independent set.