1. Show that any forest matroid is also a linear matroid. Also show that the uniform matroid $U_{2,4}$ is a linear matroid, but not a forest matroid.

Given $G$, let $A$ be the following matrix: rows indexed by vertices $v_1, \ldots, v_n$, columns indexed by edges, column $v_i v_j$ with $i < j$ has a 1 in row $v_i$, a $-1$ in row $v_j$, and a 0 else ($A$ is not quite the incidence matrix, but the incidence matrix of some directed graph obtained from $G$; it doesn’t actually matter how the edges are directed).

Then the linear matroid defined by $A$ is the same as the forest matroid defined by $G$. To see this, we’ll show that a forest in the graph corresponds exactly to an independent set of columns of $A$.

If $e_1 e_2 \cdots e_k$ is a cycle, then the corresponding columns, call them $c(e_i)$, are linearly dependent, because we can write $\sum_{i=1}^n \epsilon_i c(e_i) = 0$, for some choice of $\epsilon_i \in \{1, -1\}$. A non-forest in $G$ contains a cycle, hence the corresponding set of columns is linearly dependent.

On the other hand, suppose the set of columns corresponding to a forest is dependent, i.e. there is a linear relation $\sum a_i c(e_i) = 0$ between them. We can assume that each $a_i$ is nonzero (otherwise we remove that edge, and we still have a forest). But any nontrivial forest must have a vertex of degree 1 (a leaf), so also an edge $e_i$ which is the only one with a nonzero entry in the row corresponding to the leaf. But then $\sum a_i c(e_i)$ will have a nonzero coefficient in that row, a contradiction.

Here is a matrix whose linear matroid is exactly $U_{2,4}$:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$  

One can easily check that the linearly independent subsets of columns are exactly the subsets with $\leq 2$ elements.

Suppose $U_{2,4}$ is a forest matroid. Then the 4 1-element subsets correspond to 4 edges of the graph. The 3-element subsets are 3 edges that are not a forest, which means they must form a 3-cycle. Take one such 3-cycle; then the fourth edge must also form a 3-cycle with every 2 edges of the first 3-cycle. This is clearly not possible.

2. Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$  

Let $M$ be the corresponding linear matroid over $\mathbb{F}_2$, the field with 2 elements. Show that $M$ is not a forest matroid, and not a linear matroid over $\mathbb{R}$. ($M$ is called the Fano matroid.)

Label the columns 1 through 7.

Suppose the matroid is a forest matroid. Since $\{1, 2, 4\}$ is dependent, the corresponding edges form a 3-cycle. So does $\{1, 3, 5\}$, and its 3-cycle shares an edge with the first 3-cycle. And $\{2, 3, 6\}$ is also a 3-cycle, so 6 must connect the 2 vertices that the previous
2 3-cycles did not share. In other words, these 6 edges must form a $K_4$, a complete graph on 4 vertices. In particular, \{1, 2, 3\} forms a tree of 3 edges with a common endpoint.

But then edge 7 has nowhere to go: It would have to form a forest with any 2 of 1, 2, 3, but a non-forest with all 3 of them. This is impossible. (Note that we haven’t used $\mathbb{F}_2$, this would have worked for this matrix over any field.)

Suppose the matroid is linear over $\mathbb{R}$, so there is a set of 7 columns in some $\mathbb{R}^m$ with the same linear dependencies, and $m \geq 3$. Then 3 of these columns correspond to columns 1, 2, 3 of $A$. We can apply a linear transformation, which preserves the matroid, so that these are given by $c_1 = (1, 0, 0, \ldots)^T$, $c_2 = (0, 1, 0, \ldots)^T$, $c_3 = (0, 0, 1, \ldots)^T$. Then it follows that the columns corresponding to 4, 5, 6 of $A$ are given by $c_4 = (1, 1, 0, \ldots)^T$, $c_5 = (1, 0, 1, \ldots)^T$, $c_6 = (0, 1, 1, \ldots)^T$; for instance, the relation $1 + 2 = 4$ in $A$ implies $c_4 = c_1 + c_2$. All we’ve done here is shown that after a linear transformation the matrix over $\mathbb{R}$ should be pretty similar to $A$.

But over $\mathbb{F}_2$, the columns 4, 5, 6 are dependent, because

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 1 \\ 1 + 1 \\ 1 + 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Yet over $\mathbb{R}$ the columns $c_4, c_5, c_6$ are independent, a contradiction.

3. In a matroid $M = (X, I)$, a circuit is a minimal dependent set, i.e. a $C$ such that $C \notin I$, but if $D \subset C$ then $D \in I$.

Circuits are the generalization of cycles in graphs. The following two facts are the generalizations of standard facts about cycles.

a) Prove that if $C_1 \neq C_2$ are two circuits, and $x \in C_1 \cap C_2$, then there is a circuit $C_3 \subset (C_1 \cup C_2) \setminus \{x\}$.

b) Prove that if $Y \in I$, but $Y \cup \{x\} \notin I$, then $Y \cup \{x\}$ contains exactly one circuit.

a) Let $C_1 \neq C_2$ with $x \in C_1 \cap C_2$. There is an $x_1 \in C_1 \setminus C_2$, since $C_2$ is minimal; then also $x_1 \neq x$. Then there is a basis $B$ of $C_1 \cup C_2$ such that

$$C_1 \setminus \{x_1\} \subset B \subset C_1 \cup C_2,$$

since $C_1 \setminus \{x_1\} \in I$ but $C_1 \cup C_2 \notin I$. It follows that $x_1 \notin B$, and there is an $x_2 \in C_2$ such that $x_2 \notin B$ and $x_2 \neq x_1$. Then

$$|B| \leq |(C_1 \cup C_2) \setminus \{x_1, x_2\}| < |(C_1 \cup C_2) \setminus \{x\}|,$$

so $(C_1 \cup C_2) \setminus \{x\} \notin I$, since it is larger than $B$, so also larger than any basis of $C_1 \cup C_2$. Hence there must be a circuit in $(C_1 \cup C_2) \setminus \{x\}$.

b) Since $Y \cup \{x\} \notin I$, there must be some circuit in $Y \cup \{x\}$. Suppose there are two circuits $C_1 \neq C_2$. Then $x \in C_1 \cap C_2$, otherwise one of the circuits would be in $Y$. So by a) we have that there is a circuit $C_3 \subset (C_1 \cup C_2) \setminus \{x\}$. But this contradicts the minimality of $C_1$ (and $C_2$).
4. Describe a greedy removal algorithm for maximum weight independent sets in a matroid, which starts with $X$ and greedily removes elements. Prove that it works.

We'll assume that the weights are positive; the algorithm could just remove all non-positive elements first.

To compare, note that in a greedy removal algorithm for maximum weight forests in graphs, you would remove an edge only if the result still contained a maximal forest, because you know that a maximum weight forest would also be maximal (assuming the weights are positive). For a matroid, this corresponds to checking if the result of a removal still contains a basis.

**Greedy removal algorithm**

1. Set $Y = X$ and $S = \emptyset$;
2. Find $x \in Y \setminus S$ with minimum $w(x)$; if $Y \setminus S = \emptyset$ go to 4;
3. If $Y \setminus \{x\}$ contains a basis of $X$, set $Y := Y \setminus \{x\}$;
   set $S := S \cup \{x\}$; go to 2;
4. Return $Y$.

**Proof of correctness:** If $Y \subset X$ contains a basis, call it *good* if it contains a maximum weight basis of $X$. We'll show that the final $Y$ in the algorithm is good, which implies that it is a basis (otherwise more elements could be removed), so independent, and that it is maximum weight.

So suppose $Y$ is good, so it contains some maximum weight basis $B$ of $X$, and $x$ is chosen by the algorithm, so $Y \setminus \{x\}$ contains a basis $B'$ of $X$. By M2' we have $|B| = |B'|$. If $x \notin B$, then $B \subset Y \setminus \{x\}$, and we are done. So assume $x \in B$; we also know that $x \notin B'$. Then $|B \setminus \{x\}| < |B'|$ by M2', so by M2 there is a $y \in B' \setminus B$ such that $B'' = (B \setminus \{x\}) \cup y \in \mathcal{I}$. This implies that $B''$ is a basis, since $|B''| = |B|$.

Since $y$ is still in $Y$, it either hasn’t been considered yet, or it has already been rejected. If it had been rejected, a $Y' \supset Y$ would have occurred earlier in the algorithm, such that $Y' \setminus \{y\}$ does not contain a basis; but $B \subset Y \setminus \{y\} \subset Y' \setminus \{y\}$. So $y$ has not been considered yet, hence $w(y) \geq w(x)$. Then $w(B'') \geq w(B)$, which implies that $B''$ is also maximum, so $Y \setminus \{x\}$ is good.
5. Let $X_1, \ldots, X_m$ be a partition of $X$. Define

$$\mathcal{I} = \{ Y \subset X : \forall i \ |Y \cap X_i| \leq 1 \}.$$  

Prove that $(X, \mathcal{I})$ is a matroid. Such a matroid is called a partition matroid.

Let $G = (A \cup B, E)$ be a bipartite graph. Show that the set of matchings of $G$ is an intersection of two partition matroids with $X = E$, i.e. \{matchings\} = \mathcal{I}_1 \cap \mathcal{I}_2.

\begin{align*}
\text{M1 is obvious. For M2', observe that the bases are exactly the } B & \subset X \text{ with all } |B \cap X_i| = 1. \text{ Since the } X_i \text{ are a partition, all bases have } |B| = m. \\
\text{Let } A = \{a_1, \ldots, a_k\} \text{ and } B = \{b_1, \ldots, b_l\}. \text{ Partition } E \text{ in 2 ways:} \\
E = \bigcup \delta(a_i), \quad E = \bigcup \delta(b_i).
\end{align*}

Let

$$\mathcal{I}_1 = \{ D \subset E : \forall i, \ |D \cap \delta(a_i)| \leq 1 \},$$

$$\mathcal{I}_2 = \{ D \subset E : \forall i, \ |D \cap \delta(b_i)| \leq 1 \}.$$  

Then

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{ D \subset E : \forall v \in A \cup B, \ |D \cap \delta(v)| \leq 1 \} = \{ \text{matchings} \}.$$  

Note that this is not a partition matroid, since the $\delta(v)$ for all $v$ do not form a partition. In fact, in the notes we saw that the set of matchings cannot be a matroid.