Multiple choice

For each of the questions below choose a correct answer out of the proposed options.
For each correct answer you receive 3 points, while for each incorrect answer you lose 1 point. There is exactly one correct answer for each question.

1. How many of the following statements are true: \( n^3 = o\left(\binom{n}{4}\right) \), \( \binom{n}{4} = O\left(n^5\right) \), \( e^n = o\left(\binom{n}{5}\right) \), \( \binom{n}{4} = O\left(\binom{n}{4}\right) \).
   □ 1   □ √   □ 3   □ 4

2. The number of functions \( f : \{1, 2, \ldots, 10\} \rightarrow \{1, 2, 3\} \) is
   □ \(3^{10}\)   □ \(\binom{10}{3}\)   □ \(\binom{11}{3}\)   □ \(10^3\)

3. Let \( G \) be a graph on \( n \) vertices with \( n \) edges. Which of the following statements cannot be true?
   □ \( G \) is a connected graph.
   □ \( G \) has a cycle.
   □ \( G \) is a disconnected graph.
   □ √ \( G \) is a tree.

4. How many permutations does the set \( \{1, 2, 3, 4, 5, 6\} \) have such that 1 and 5 are not adjacent?
   □ 120   □ 240   □ √ 480   □ 720
5. Let \( n \geq 2 \) and \( P \) be the partially ordered set \((\{1, 2, 3, \ldots, 2^n\}, \prec)\) where \( a \prec b \) if \( a \) divides \( b \). Then the least number of antichains in \( P \) that cover the set \( \{1, 2, 3, \ldots, 2^n\} \) is

- \( 2^{n-2} \)
- \( \binom{n}{2} \)
- \( n \)
- \( n+1 \)

6. The number of trees on \( n \) labeled vertices such that the corresponding Prüfer code is a strictly monotone sequence is

- \( \sqrt{n(n-1)} \)
- \( 0 \)
- \( 2^{n-2} \)
- \( \binom{n}{n-2} \)

7. For every integer \( n \geq 5 \), the maximum number of distinct subsets \( A_1, A_2, \ldots \) that can be selected from the set \( \{1, 2, \ldots, n\} \) with the property that \( |A_i \cap A_j| = n-2 \) for any \( i \neq j \), is equal to

- \( 2 \)
- \( 4 \)
- \( n \)
- \( \binom{n}{2} \)

8. How many labeled trees are there on vertex set \( \{1, \ldots, n\} \) such that both the vertices \( n \) and \( n-1 \) have degree 1?

- \( n^{n-4} \)
- \( (n-1)^{n-3} \)
- \( (n-2)^{n-4} \)
- **None of the above**

9. The number of integers in \( \{1, \ldots, 60\} \) that are divisible by at least one of the numbers 2, 3, 5 is

- 45
- **44**
- 55
- 54

10. Which of the following graphs has more perfect matchings?

- A tree on 20 vertices
- **A cycle on 20 vertices**
- A cycle on 19 vertices
- A complete graph on 11 vertices
True-false questions
For each of the statements below decide whether they are true or false. For each correct answer you receive 2 points, while for each incorrect answer you lose 2 points.

1. If $n$ is a positive integer with $k$ prime factors, then $\phi(n)$ is divisible by $2^k$.
   □ True
   √ False

2. Every graph on $n \geq 2$ vertices with more than $\binom{n-1}{2}$ edges is connected.
   √ True
   □ False

3. Let $e$ be an edge of minimum weight in the connected weighted graph $G$. Every minimum spanning tree of $G$ contains $e$.
   □ True
   √ False

4. $2n^2 + 1 = O(n \log n)$ (as $n \to \infty$).
   □ True
   √ False

5. Let $d \geq 1$ and suppose every vertex of a bipartite graph $G = (A \cup B, E)$ has degree $d$. Then $G$ has a perfect matching $M$. (In other words, every vertex of $A \cup B$ belongs to an edge of $M$)
   √ True
   □ False

6. The number of increasing functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is equal to the number of solutions of $\sum_{i=1}^{m} x_i = n + 1$ satisfying $x_1, x_2, \ldots, x_m \in \{0, 1, 2, \ldots, n + 1\}$.
   □ True
   √ False

7. Every sequence of $mn - 1$ distinct numbers contains an increasing sequence of length $n - 1$ or a decreasing sequence of length $m - 1$.
   √ True
   □ False

8. The maximum number of subsets of $\{1, 2, \ldots, 2017\}$, none of which contains another, is strictly less than the number of 1009-element subsets of $\{1, 2, \ldots, 2018\}$.
   √ True
   □ False

9. If a finite partially ordered set has an antichain of size 17, then it can be covered by 17 chains.
   □ True
   √ False

10. There is a graph on ten vertices whose degrees are 9, 8, 8, 8, 6, 5, 4, 4, 2, 2.
    □ True
    √ False
Problems
Give solutions to the following problems. For each correct solution you receive 6 points. For each problem you have to explain your answer, that is to give a complete proof.

1. How many solutions does the inequality \( x_1 + x_2 + \cdots + x_k \leq n \) have assuming that each \( x_i \) is a non-negative integer? Justify your answer.

We define the auxiliary variable \( x_{k+1} := n - (x_1 + x_2 + \cdots + x_k) \). By the assumption, we have \( x_{k+1} \geq 0 \). So the problem is equivalent to finding the number of solutions of

\[
x_1 + x_2 + \cdots + x_k + x_{k+1} = n
\]

in the set of nonnegative integers. By what one has seen in the class, the number of such solutions is \( \binom{n+(k+1)-1}{k+1-1} = \binom{n+k}{k} \).
2. Find the tree corresponding to the following Prüfer code: $(1, 2, 1, 2, 9, 1, 3, 4, 1, 5, 8, 1)$.

Solution.
3. Find the number of sequences \((a_1, a_2, a_3, a_4, a_5)\) such that \(a_i \in \{1, 2, \ldots, 5\}\) for every \(i\), and there is no \(j\), \(1 \leq j \leq 3\), such that \(a_j = a_{j+1} = a_{j+2}\).

Let \(A_1\) be the set of sequences \((a_1, a_2, a_3, a_4, a_5)\) such that \(a_1 = a_2 = a_3\), let \(A_2\) be the set of such sequences such that \(a_2 = a_3 = a_4\) and let \(A_3\) be the set of such sequences such that \(a_3 = a_4 = a_5\). Then \(A_1 \cup A_2 \cup A_3\) is the set of bad sequences.

There are \(5^5\) sequences in total, so the answer will be \(5^5 - |A_1 \cup A_2 \cup A_3|\). To compute the latter, we use inclusion-exclusion:

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|.
\]

Here \(|A_1| = |A_2| = |A_3| = 5^3\) because in these sets there are sequences with 3 identical values, so we have to make 3 choices of the values from \(\{1, \ldots, 5\}\).

\(A_1 \cap A_2\) is the set of sequences such that \(a_1 = \cdots = a_4\), so there are two choices to make: the value of these for elements and the value of \(a_5\). Hence \(|A_1 \cap A_2| = 5^2\). Similarly, \(A_2 \cap A_3\) is the set of sequences such that \(a_2 = \cdots = a_5\), so \(|A_2 \cap A_3| = 5^2\).

\(A_1 \cap A_3 = A_1 \cap A_2 \cap A_3\) is the set of constant sequences, so \(|A_1 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 5\).

Altogether, we get that the number of sequences satisfying the conditions is

\[
5^5 - 3 \cdot 5^3 + 2 \cdot 5^2 + 5 - 5 = 5^5 - 3 \cdot 5^3 + 2 \cdot 5^2.
\]

(The numerical value is 2800.)
4. Find a minimum spanning tree for the following graph. Mark it clearly on one of the figures (you can use the other figure as draft). Name and describe the algorithm you have used.

A minimum spanning tree is marked by red in the following picture:

In order to find a minimum spanning tree of a graph one can use either Kruskal’s algorithm or Prim’s algorithm (one is enough); Weight of minimum spanning tree = 35.

Short description of Kruskal’s algorithm:
- Sort all the edges in non-decreasing order of their weight.
- Pick the smallest edge. Check if it forms a cycle with the spanning tree formed so far. If cycle is not formed, include this edge. Else, discard it.
- Repeat step until there are no edges left.

Short description of Prim’s algorithm:
- Initialize a tree with a single vertex, chosen arbitrarily from the graph.
- Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.
5. Let $S$ be a sequence of $n$ (not necessarily distinct) integers. Assume $n > pqr$, where $p$, $q$, $r$ are three positive integers. Prove that there exists either a strictly increasing subsequence of size $p + 1$, or a strictly decreasing subsequence of size $q + 1$ or a subsequence of size $r + 1$ consisting of the same integer.

Denote the sequence by $S$. If there exists an element of $S$ which is repeated at least $r + 1$ times, then we get a constant subsequence of size at least $r + 1$. So suppose every element is repeated at most $r$ times in $S$. Now for every element appearing more than once, remove all the other copies of the element from $S$ so that at the end we get a sequence $S'$ consisting of distinct numbers. Since each element appeared at most $r$ times in $S$, we deduce that $S'$ has at least $pq + 1$ distinct elements. We show that $S'$ has either an increasing subsequence of $p + 1$ elements or a decreasing subsequence of $q + 1$ elements which correspond to an increasing or a decreasing subsequence in $S$.

Let $S' = (x_1, x_2, ..., x_{pq+k})$, where $k \geq 1$, and by contradiction, suppose it has neither any increasing subsequences of length $p+1$ nor any decreasing subsequences of length $q+1$. Define the partial order $\preceq$ on $S'$ so that $x_i \preceq x_j$ whenever $i \leq j$ and $x_i \leq x_j$ (as real numbers). One can check that under this partial ordering, increasing subsequences of $S'$ correspond to chains, while decreasing subsequences of $S'$ correspond to antichains. Since we assumed that every antichain in $S'$ has size at most $q$, by Dilworth’s theorem we deduce that there is a covering of the elements of $S'$ by at most $q$ chains, each of size at most $p$, and thus $S'$ has at most $pq$ elements, which is a contradiction.
6. **[Bonus problem]** Let \( n, k \) be positive integers. Prove that at least one of the numbers

\[
\binom{n}{k}, \binom{n+1}{k}, \ldots, \binom{n+k}{k}
\]

is not divisible by 7.

First, note that for \( i = 0, 1, \ldots, k - 1 \), we have

\[
\binom{n+i}{k-1} = \binom{n+i+1}{k} - \binom{n+i}{k}.
\]

Therefore, if by contradiction, all the binomial coefficients

\[
\binom{n}{k}, \binom{n+1}{k}, \ldots, \binom{n+k}{k}
\]

are divisible by 7, then we get that all the following binomial coefficients are also divisible by 7:

\[
\binom{n}{k-1}, \binom{n+1}{k-1}, \ldots, \binom{n+k-1}{k-1}.
\]

Continuing this procedure, we get that \( \binom{n}{0} \) is also divisible by 7, which is a contradiction. (since \( \binom{n}{0} = 1 \))