Discrete mathematics - optional midterm exam
November 20, 2014.

Multiple choice (there is exactly one correct answer)
For every correct answer you receive 2 points, while for every incorrect answer you lose 1 point.

1. The number of subsets of an \((n + k)\)-element set is
   - \(\binom{n+k}{k}\)
   - \(2^{n+k}\)
   - \(3^{n+k}\)
   - \((n + k)!\)

2. Let \(n, k\) be positive integers. The statement \(\binom{n}{k} \leq \binom{n}{k+1}\) is
   - always true
   - true for some values of \(n\) and \(k\)
   - never true

3. How many of the following four statements are true:
   \[\sum_{k=1}^{n} k^2 = O(n), \sum_{k=1}^{n} k^2 = O(n^2), \sum_{k=1}^{n} k^2 = O(n^3), \text{ and } \sum_{k=1}^{n} k^2 = O(n^4)?\]
   - none
   - one
   - two
   - at least three

4. Let \(f(n)\) denote the number of subsets of an \(n\)-element set. How many of the following four relations are true?
   \(f(n) = O(n^2), f(n) = O(\log n), f(n) = O(5^n), \text{ and } f(n) = O(1/2^n)\)?
   - none
   - one
   - two
   - at least three

5. Which of the following is the largest?
   - \(\binom{11}{5}\)
   - \(\binom{11}{7}\)
   - \(\binom{11}{3}\)
   - \(\binom{11}{2}\)

6. Let \(G\) be a tree, and \(p, q\) two vertices of \(G\). Then
   - there is a unique cycle that contains \(p\) and \(q\)
   - there may be two cycles containing \(p\) and \(q\)
   - there is no cycle containing \(p\) and \(q\)
True-false questions

For every correct answer you receive 2 points, while for every incorrect answer you lose 2 points.

Which of the following statements is true, which one false?

1. Every tree is a bipartite graph.
   □ True
   □ False

2. Let $G$ be a bipartite graph with two equal parts. If there is a cycle in $G$ that covers all vertices, then there is a perfect matching in $G$.
   □ True
   □ False

3. Consider the following relation on the set of integers: we say that $a \leq b$ if $a$ divides $b$. Then $\mathbb{Z}$ with this relation a poset.
   □ True
   □ False

4. Let $X = \{1, 2, 3, 4, 5\}$, and $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{1, 5\}, \{2, 5\}\}$. Consider the usual partial ordering on the set of all subsets of $X$, that is, for any $X_1, X_2 \subseteq X$ we have $X_1 \leq X_2$ if and only if $X_1 \subseteq X_2$. Then $\mathcal{F}$ is an antichain.
   □ True
   □ False

5. Let $T$ be a tree on $n$ vertices. Assume that the number of leaves of $T$ is at most $n - 2$. Then the Prüfer code of $T$ has at least two distinct numbers.
   □ True
   □ False

6. Let $f(n)$ be the number of spanning trees on $n$ labeled vertices. Is it true that $f(n) = O(2^n)$?
   □ True
   □ False

7. If we can cover a finite partially ordered set by 10 chains, then it has an antichain of size 10.
   □ True
   □ False
Problems

For every correct solution you receive 6 points. For each problem you have to explain your answer, that is give a complete proof. The solution to the Bonus Problem can also be handed in next time as a starred problem, for the context.

1. What is the number of positive integers less than 462, which are not divisible by any of the numbers 3, 7, or 11?

2. Find a minimum weight spanning tree for the following graph. Name or give a short description of the algorithm you have used.
3. Does the following bipartite graph have a perfect matching?

4. Let $S$ be a set of 10 distinct points that lie inside the square $[0, 2] \times [0, 2] = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 2\}$. Let $f : [0, 2] \times [0, 2] \rightarrow [0, 2] \times [0, 2]$ be a mapping defined as $f(x, y) = (\lfloor x \rfloor, \lfloor y \rfloor)$. Prove that $S$ has two elements $(x_1, y_1), (x_2, y_2)$ such that $f(x_1, y_1) = f(x_2, y_2)$. 
5*. **Bonus problem for extra credit**

Is the following statement true? The maximum number of subsets of an \( n \)-element set with the property that any two of them have at least 2 elements in common is \( 2^{n-2} \).