1. Let \( k \geq 2 \). Show that every \( k \)-connected graph with at least \( 2k \) vertices contains a cycle of length at least \( 2k \).

If \( G \) is \( k \)-connected, then \( \delta(G) \geq k \), so by Proposition 1.2.2, \( G \) contains a cycle of length at least \( \delta(G) + 1 \geq k \). Let \( C \) be a longest cycle in \( G \), and suppose its length is less than \( 2k \). So we have \( k \leq |V(C)| < 2k \). Since \( |V(G)| \geq 2k \), there is a vertex \( x \notin V(C) \).

By Lemma 8.2.2 from the notes (or by a similar application of Menger’s Theorem), and using the fact that \( |V(C)| \geq k \), there are \( k \) paths from \( x \) to \( V(C) \) that are disjoint except for \( x \), with each path containing only one vertex of \( C \). By the pigeonhole principle and the fact that \( |V(C)| < 2k \), there must be two of these paths that end at adjacent vertices \( y, z \) of \( V(C) \). Then replacing the edge \( yz \) in \( C \) by the path from \( y \) to \( x \) to \( z \), we get a new cycle. It is longer, since each of the two paths has at least one edge. This contradicts the choice of \( C \).

2. Show that any two vertices in a 3-connected graph are connected by two internally disjoint paths of different lengths.

Let \( x, y \) be two vertices in a 3-connected graph. There are three internally disjoint paths \( P, Q, R \) between \( x \) and \( y \); if two of them have different lengths we are done, so assume \( P, Q, R \) have the same length \( L \). At most one of them could be a single edge, so we have \( L \geq 2 \). Hence \( V(P) \setminus \{x, y\} \) and \( V(Q) \setminus \{x, y\} \) are not empty.

Let \( p \) and \( q \) be two vertices on two of these paths, not equal to \( x \) or \( y \). There are three internally disjoint paths from \( p \) to \( q \); at least one of these paths does not pass through \( x \) or \( y \). So for any \( p \) from one of \( P, Q, R \), and \( q \) from one of the other two, there is a path between \( p \) and \( q \) that does not pass through \( x \) or \( y \). Let \( S \) be the shortest path between any such \( p \) and \( q \); without loss of generality, \( S \) goes from \( r \in V(P) \setminus \{x, y\} \) to \( s \in V(Q) \setminus \{x, y\} \). By minimality, \( S \) does not contain any vertex of \( P \) or \( Q \) other than \( r \) and \( s \), and \( S \) does not contain any vertex of \( R \).

We now have two new paths from \( x \) to \( y \) that are internally disjoint from \( R \): One goes from \( x \) along \( P \) to \( r \), then along \( S \) to \( s \), and then along \( Q \) to \( y \); the other goes from \( x \) along \( Q \) to \( s \), along \( S \) to \( r \), and along \( P \) to \( y \). The total length of these two paths is \( 2L + 2|E(S)| \), so one of the two must have length different from \( L \).

3. Prove that a \( k \)-connected graph \( G \) has \( |E(G)| \geq \frac{1}{2}k|V(G)| \). For even \( k \), find a \( k \)-connected graph \( G \) with \( |E(G)| = \frac{1}{2}k|V(G)| \).

The lower bound follows from \( \delta(G) \geq k \), since \( |E(G)| = \frac{1}{2} \sum d(v) \geq \frac{1}{2}k|V(G)| \).

For the construction, start with a cycle and connect each vertex to the \( k/2 \) nearest neighbors on each side. That gives the right number of edges. To see that it is \( k \)-connected, remove \( k - 1 \) vertices and consider two vertices \( x, y \). One of the two half-cycles between \( x \) and \( y \) has less than \( k/2 \) removed vertices. Then along that half-cycle we can find a path from \( x \) to \( y \), because if we travel in one direction, every vertex has \( k/2 \) neighbors in that direction, not all of which can have been removed.
4. A graph \( G \) is \( k \)-edge-connected if for every \( S \subset E(G) \) of size \( k - 1 \) the graph \( G - S \) is connected. Show that if \( G \) is \( k \)-connected, then \( G \) is \( k \)-edge-connected. Give an example to show that the converse is not true.

Suppose \( G \) is \( k \)-connected, and let \( F \) be a minimal set of edges such that \( G - F \) is disconnected. If some vertex \( v \) of \( G \) is not incident to \( F \), then let \( C \) be the component of \( G - F \) that contains \( v \). Every edge of \( F \) has at most one vertex in \( C \), by minimality of \( F \). Taking all the endpoints of edges of \( F \) that lie in \( C \), we get at most \( |F| \) vertices that disconnect \( v \) from some other vertex. So \( |F| > k - 1 \).

Otherwise, every vertex is incident with an edge of \( F \). For any vertex \( v \), \( N(v) \) disconnects the graph (unless it is complete, in which case it is \( k \)-edge-connected). Every \( w \in N(v) \) is either connected to \( v \) by an edge of \( F \), or is incident to a distinct edge of \( F \) (if for \( w, w' \in N(v) \) we have \( ww' \in F \), it would contradict minimality of \( F \)). So \( k - 1 < |N(v)| \leq |F| \).

Take two complete graphs sharing one vertex. It is highly edge-connected but only \( 1 \)-connected.

5. Use Menger’s Theorem to reprove König’s Theorem.

Let \( G \) be a bipartite graph with bipartition \( V(G) = A \cup B \). A matching of size \( m \) corresponds to a set of \( m \) disjoint \( AB \)-paths (in the sense of Menger’s Theorem). A vertex cover corresponds to a set of vertices that cover every original edge, which means that it cuts every \( AB \)-path, so it is an \( AB \)-separator. By Menger’s Theorem, the maximum size of a set of disjoint \( AB \)-paths equals the minimum size of an \( AB \)-separator.

If you prefer the version of Menger with two vertices, add a vertex \( a \) connected to all vertices in \( A \), and a vertex \( b \) connected to all vertices in \( B \). Then a matching corresponds to a set of internally disjoint \( ab \)-paths.

6. Show that in a 3-connected graph, any two longest cycles share at least 3 vertices.

Suppose \( C_1 \) and \( C_2 \) are two longest cycles (it doesn’t matter if we interpret this as both being the same length, or as one being longest and the other second-longest).

If \( C_1 \) and \( C_2 \) are disjoint, then Menger’s Theorem gives three disjoint paths between \( V(C_1) \) and \( V(C_2) \). If \( C_1 \) and \( C_2 \) share exactly one vertex \( x \), then Menger gives two disjoint paths between \( V(C_1) \) and \( V(C_2) \) (not counting the one-vertex path \( x \)). Either way, we have two disjoint paths \( P, Q \) between \( C_1 \) and \( C_2 \); let’s say \( P \) goes from \( p_1 \in V(C_1) \) to \( p_2 \in V(C_2) \), and that \( Q \) goes from \( q_1 \in V(C_1) \) to \( q_2 \in V(C_2) \).

In the case where \( C_1 \) and \( C_2 \) are disjoint, we get a longer cycle by going from \( p_1 \) to \( q_1 \) along the longer part of \( C_1 \), from \( q_1 \) to \( q_2 \) along \( Q \), from \( q_2 \) to \( p_2 \) along the longer part of \( C_2 \), and finally from \( p_2 \) to \( p_1 \) along \( P \).

When \( C_1 \) and \( C_2 \) share one vertex \( x \), this does not work, and we do the following. Let \( C_3 \) be the following cycle: Go from \( p_1 \) to \( q_1 \) along the part of \( C_1 \) not containing \( x \), from \( q_1 \) to \( q_2 \) along \( Q \), from \( q_2 \) to \( p_2 \) along the part of \( C_2 \) containing \( x \), and finally from \( p_2 \) back to \( p_1 \) along \( P \). Let \( C_4 \) be the following cycle: Go from \( p_2 \) to \( q_2 \) along the part of \( C_2 \) not containing \( x \), from \( q_2 \) to \( q_1 \) along \( Q \), from \( q_1 \) to \( p_1 \) along the part of \( C_1 \) containing \( x \), and finally from \( p_1 \) back to \( p_2 \) along \( P \). Now we have \( |E(C_3)| + |E(C_4)| = |E(C_1)| + |E(C_2)| + |E(P)| + |E(Q)| \), so one of \( C_3, C_4 \) is longer than \( C_1 \) or \( C_2 \).

Now suppose \( C_1 \) and \( C_2 \) share exactly two vertices. Menger gives one path \( P \) from \( V(C_1) \) to \( V(C_2) \). We won’t spell it out, but again we can find two new cycles \( C_3, C_4 \) with \( |E(C_3)| + |E(C_4)| = |E(C_1)| + |E(C_2)| + |E(P)| \), which shows that one of \( C_3, C_4 \) must be longer than \( C_1 \) or \( C_2 \).
*7. Prove that a graph $G$ with $|E(G)| \geq 2k|V(G)|$ contains a $k$-connected subgraph.

We use induction on $|V(G)|$ with the following strengthened induction claim:

If $|V(G)| \geq 2k$ and $|E(G)| \geq 2k|V(G)| - 2k^2$, then $G$ has a $k$-connected subgraph.

This finishes the problem, since if $|E(G)| \geq 2k|V(G)|$, then $\binom{|V(G)|}{2} \geq 2k|V(G)|$, which implies $|V(G)| \geq 4k + 1 > 2k$, so the induction claim applies.

First note that if $|V(G)| = 2k$, then $2k|V(G)| - 2k^2 = 2k^2 > \binom{2k}{2} = \binom{|V(G)|}{2}$, so we cannot have this many edges, and the claim trivially holds. So suppose we have $G$ with $|V(G)| > 2k$ and $|E(G)| \geq 2k|V(G)| - 2k^2$.

If $G$ has a vertex $v$ of degree less than $2k$, then we can remove $v$ and apply induction, since

$$|E(G - v)| > |E(G)| - 2k \geq 2k|V(G)| - 2k^2 - 2k = 2k(|V(G)| - 1) - 2k^2 = 2k|V(G - v)| - 2k^2.$$ 

Thus we can assume that every vertex of $G$ has degree at least $2k$.

If $G$ is $k$-connected, then we are done. Otherwise, there is a set $X$ with $|X| = k - 1$ that disconnects $G$. Thus we can get two subgraphs $G_1$ and $G_2$ with $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = X$, and $E(G_1) \cup E(G_2) = E(G)$. For instance, let $G_1$ be the union of $X$ with a component of $G - X$, and let $G_2$ be the union of $X$ with the other components of $G - X$.

Since each $G_i$ contains a vertex not in $X$, and that vertex has degree at least $2k$, we have $|V(G_i)| \geq 2k$ for both $i$. We cannot have $|E(G_i)| < 2k|V(G_i)| - 2k^2$ for both $i$, since then

$$|E(G)| \leq |E(G_1)| + |E(G_2)| < (2k|V(G_1)| - 2k^2) + (2k|V(G_2)| - 2k^2) = 2k(|V(G_1)| + |V(G_2)| - 2k) = 2k(|V(G)| + k - 1 - 2k) < 2k|V(G)| - 2k^2.$$ 

Thus the induction claim applies to at least one of the $G_i$, which gives a $k$-connected subgraph.