1. Prove that a k-regular bipartite graph (with \( k \geq 1 \)) has a perfect matching.

Let \( G \) be a k-regular bipartite graph with bipartition \( V(G) = A \cup B \). First note that \( |A| = |B| \), since
\[
|A| = \sum_{a \in A} \deg(a) = |E| = \sum_{b \in B} \deg(b) = |B|.
\]

Using Hall’s Theorem: Hall’s Theorem tells us there is matching that matches \( A \) if \( |N(S)| \geq |S| \) for all \( S \subseteq A \). The number of edges from \( S \) to \( N(S) \) is \( k|S| \), and this is the most the number of edges from \( N(S) \) to \( A \), which is \( k|N(S)| \). So we have \( k|S| \leq k|N(S)| \), which implies \( |N(S)| \geq |S| \).

Using König’s Theorem: We observe that any vertex cover \( C \) has at least \( |A| \) vertices, since each vertex of \( C \) covers exactly \( k \) of the \( k|A| \) edges. Then König’s Theorem tells us that a maximum matching has at least \( |A| \) edges, which is only possible if it has exactly \( |A| = |B| \) edges, so is perfect.

2. Show that a tree has at most one perfect matching.

We use induction on \( |V(T)| \). For \( |V(T)| \leq 2 \) the statement is trivial. Let \( T \) be a tree on at least 3 vertices. By a lemma from class, \( T \) has a leaf \( u \), attached to some vertex \( v \). Any perfect matching of \( T \) must contain \( uv \), since that is the only way to match \( u \). Let \( F \) the graph obtained by removing \( u, v \) and every incident edge; \( F \) is a forest. Each connected component of \( F \) is a tree, so by induction has at most one matching, which implies that \( F \) has at most one matching. Any matching of \( T \) must consist of the edge \( uv \) and a matching of \( F \), so it follows that \( T \) has at most one matching.

Alternative solution: Suppose a tree has two perfect matchings, \( M \) and \( M’ \). Consider the symmetric difference \( M \triangle M’ \) as a subgraph \( D \) on \( V(T) \). Every vertex has degree 0 or 2 in \( D \), so \( D \) must be a union of cycles and isolated vertices (by a problem from Problem Set 1). But a tree contains no cycles, so \( D \) only consists of isolated vertices. That implies that \( M \) and \( M’ \) are the same.

3. Show that a maximal matching is at least half the size of a maximum matching.

Let \( M \) be a maximal matching and \( N \) a maximum matching. Suppose \( |M| < |N|/2 \). Then the number of vertices in edges of \( M \) is strictly less than the number of edges in \( N \), so there must be an edge \( e \in N \) with neither endpoint in an edge of \( M \). But then we could add \( e \) to \( M \), contrary to it being maximal.

4. Prove that any bipartite graph \( G \) has a matching of size at least \( |E(G)|/\Delta(G) \).

By König’s Theorem, it suffices to prove that a vertex cover of \( G \) cannot have fewer than \( |E(G)|/\Delta(G) \) vertices. This follows directly from the fact that every vertex covers at most \( \Delta(G) \) edges.

5. Show that a bipartite graph \( G \) has a perfect matching if and only if its largest independent set has size \( |V(G)|/2 \) (i.e., a subset of \( |V(G)|/2 \) vertices, no two of which are adjacent).

By König’s Theorem, \( G \) has a perfect matching if and only if its minimum vertex cover has size \( |V(G)|/2 \). Note that \( C \subset V(G) \) is a vertex cover if and only if \( V(G) \setminus C \) is independent. Thus if \( c \) is the size of a minimum vertex cover, then \( |V(G)| - c \) is the size of a maximum independent set. Combining these statements proves the claim.
6. Show that if $G$ is a bipartite graph with $|N(S)| \geq |S| - d$ for all $S \subset V(G)$, then $G$ has a matching with $\frac{1}{2}|V(G)| - d$ edges.

Let $V(G) = A \cup B$ be a bipartition of $G$, with $|A| \geq |B|$. Add $d$ new vertices to $B$, each connected to all vertices in $A$; let $G'$ be the new graph. Then $G'$ has $|N_{G'}(S)| \geq |S|$ for every $S \subset A$ ($S$ has at least $|S| - d$ neighbors from $G$, and is connected to the $d$ new vertices). By Hall's Theorem, $G'$ has a matching for $A$, which has $|A| \geq (|A| + |B|)/2 = |V(G)|/2$ edges. At most $d$ of these edges contain a new vertex of $G'$, which leaves at least $|V(G)|/2 - d$ edges from $G$.

7. An $r \times s$ Latin rectangle is an $r \times s$ matrix $A$ with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a Latin square. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

Define a bipartite graph whose vertex set consists of two copies of $\{1, \ldots, s\}$, call them $S_1$ and $S_2$. We connect $i \in S_1$ with $j \in S_2$ if the $i$-th column of the $r \times s$ Latin rectangle does not contain the number $j$. What we are looking for is a matching that matches $S_1$, since then we can put numbers on row $r + 1$ such that no number is repeated in that row, and no number is repeated in a column.

To see if such a matching exists we use Hall’s Theorem, or more specifically Problem 1 above. A column $i \in S_1$ contains $r$ distinct numbers, so there are $s - r$ numbers that it does not contain. That means that the vertex $i \in S_1$ has degree $s - r$. On the other hand, a number $j \in S_2$ occurs exactly once in each of the $r$ rows, and at most once in any of the $s$ columns. Hence there are $s - r$ columns that do not contain $j$, so the degree of $j \in S_2$ is $s - r$. Therefore, the graph is $(s - r)$-regular, so by Problem 1, there is a perfect matching.

8. Consider the following game on a bipartite graph $G$. Player 1 picks any vertex $v_1$. Player 2 then has to pick $v_2$ to be a neighbor of $v_1$ that was not picked before, then Player 1 picks $v_3$ to be a neighbor of $v_2$ that was not picked before, etc. Thus the players build a path $v_1v_2v_3 \cdots$. The last player that is able to pick a vertex is the winner.

Prove that Player 2 has a winning strategy if $G$ has a perfect matching, while otherwise Player 1 has a winning strategy.

If $G$ has a perfect matching $M$, then Player 2 can use the following strategy. If Player 1 picks vertex $u$, Player 2 picks the other endpoint of the edge of $M$ that matches $u$. By the definition of a perfect matching, Player 2 can always choose in this way. Because the matching is perfect, Player 2 will be the last to pick. So this is a winning strategy.

If $G$ does not have a perfect matching, then Player 1 can use the following strategy. Let $M$ be a maximum matching. Player 1 picks a vertex $u$ that is unmatched by $M$. Whichever neighbor $v$ Player 2 picks, it must be matched by some edge of $M$, since otherwise $uv$ could be added to $M$. Then Player 1 picks the other endpoint of the edge that matches $v$. Continue like this. Player 1 always picks the other endpoint of the edge that matches the choice of Player 2. This is always possible, because if the vertex chosen by Player 2 were unmatched, there would be an augmenting path for $M$, which is not possible since $M$ is maximum.
*9. Prove the following statement using Hall’s Theorem (and really using it). Let $X$ be a finite set and $S$ a set of subsets of $X$, such that there are no distinct $S, T \in S$ with $S \subset T$. Then

$$|S| \leq \left( \frac{|X|}{\lfloor |X|/2 \rfloor} \right).$$

Let $\mathcal{P}(X)$ be the set of all subsets of $X$, and write $\mathcal{P}_k(X)$ for the set of all subset of $X$ with $k$ elements. We view $\mathcal{P}(X)$ as a graph, with an edge between $S$ and $T$ if $|S \Delta T| = 1$ (in other words, one set is the other plus another element).

Consider $\mathcal{P}_k(X)$ and $\mathcal{P}_{k+1}(X)$ for some $k < |X|/2$. They form a bipartite graph, with an edge between $S \in \mathcal{P}_k(X)$ and $T \in \mathcal{P}_{k+1}(X)$ if $S \subset T$. By Hall’s Theorem, this graph has a matching that matches $\mathcal{P}_k(X)$ if for every $U \subseteq \mathcal{P}_k(X)$ we have $|N(U)| \geq |U|$. To check this, we double count as follows. A set $S \in \mathcal{P}_k(X)$ is adjacent to $|X| - k$ sets $T \in \mathcal{P}_{k+1}(X)$, one for each element not in $S$. On the other hand, a set $T \in \mathcal{P}_{k+1}(X)$ is adjacent to $k + 1$ sets $S \in \mathcal{P}_k(X)$, one for each element in $T$. This gives

$$|(X| - k)|U| \leq (k + 1)|N(U)| \implies |N(U)| \geq \frac{|X| - k}{k + 1}|U|.
$$

For $k < |X|/2$, the fraction is at least 1, which verifies Hall’s condition. We can do the same thing for $k > |X|/2$, which gives a matching from $\mathcal{P}_{k+1}(X)$ to $\mathcal{P}_k(X)$.

Combining the edges of all these matchings gives a subgraph whose connected components are disjoint paths. Every vertex of $\mathcal{P}(X)$ is in one of these paths, and each path contains a vertex in $\mathcal{P}_{\lfloor |X|/2 \rfloor}(X)$, so there are exactly $\left( \frac{|X|}{\lfloor |X|/2 \rfloor} \right)$ paths. The paths have the property that for any edge $ST$ of a path, we have $S \subset T$ or $T \subset S$. Hence the set $S$ from the question contains at most one set from each path, which implies that $|S| \leq \left( \frac{|X|}{\lfloor |X|/2 \rfloor} \right)$.

**Note:** This statement is known as Sperner’s Theorem and has many different proofs.

*10. Deduce Hall’s Theorem from König’s Theorem, and deduce König’s Theorem from Hall’s Theorem. (For both theorems, your proof should really use the other theorem to obtain a relatively simple proof.)

**Hall from König:** Suppose there is no matching that matches $A$. Then by König’s Theorem there is a vertex cover $C$ with $|C| < |A|$ vertices. Consider $C \cap A$ and $C \cap B$. There is no edge between $A \setminus C$ and $B \setminus C$, so $N(A \setminus C) \subset C \cap B$. But then

$$|N(A \setminus C)| \leq |C \cap B| = |C| - |C \cap A| < |A| - |C \cap A| = |A \setminus C|,$$

which violates Hall’s condition.

**König from Hall:** Let $C$ be a minimum cover; we show that $G$ has a matching of size $|C|$. Again consider $C \cap A$ and $C \cap B$. Since there are no edges between $A \setminus C$ and $B \setminus C$, every edge either goes from $C \cap A$ to $B$ or from $C \cap B$ to $A$. We can define two disjoint bipartite subgraphs of $G$, a graph $H_1$ with vertex sets $C \cap A$ and $B \setminus C$, and a graph $H_2$ with vertex sets $C \cap B$ and $A \setminus C$. We will show that $H_1$ has a matching that matches $C \cap A$ and $H_2$ has a matching that matches $C \cap B$, so these combine into a matching of size $|C|$, which finishes the proof.

Consider $H_1$ and let $D \subset C \cap A$. If $|N_{H_1}(D)| < |D|$, then we could replace $D \subset C$ by $N_{H_1}(D)$ to get a smaller vertex cover, contradicting $C$ being minimum. Indeed, if an edge $e$ is covered by $v \in D$, then either it is in $H_1$ and also covered by a vertex in $N_{H_1}$, or it is not in $H_1$ and incident to a vertex in $C \cap B$, so it is covered anyway. Thus Hall’s condition holds for $H_1$ and there is a matching in it that matches $C \cap A$. Similarly, we get a matching in $H_2$ that matches $C \cap B$. 