1. Prove that a $k$-regular bipartite graph (with $k \geq 1$) has a perfect matching.

2. Show that a tree has at most one perfect matching.

3. Show that a maximal matching is at least half the size of a maximum matching.

4. Prove that any bipartite graph $G$ has a matching of size at least $|E(G)|/\Delta(G)$.

5. Show that a bipartite graph $G$ has a perfect matching if and only if its largest independent set has size $|V(G)|/2$ (i.e., a subset of $|V(G)|/2$ vertices, no two of which are adjacent).

6. Show that if $G$ is a bipartite graph with $|N(S)| \geq |S| - d$ for all $S \subset V(G)$, then $G$ has a matching with $\frac{1}{2}|V(G)| - d$ edges.

7. An $r \times s$ Latin rectangle is an $r \times s$ matrix $A$ with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a Latin square. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

8. Consider the following game on a bipartite graph $G$. Player 1 picks any vertex $v_1$. Player 2 then has to pick $v_2$ to be a neighbor of $v_1$ that was not picked before, then Player 1 picks $v_3$ to be a neighbor of $v_2$ that was not picked before, etc. Thus the players build a path $v_1v_2v_3\cdots$. The last player that is able to pick a vertex is the winner.

Prove that Player 2 has a winning strategy if $G$ has a perfect matching, while otherwise Player 1 has a winning strategy.

*9. Prove the following statement using Hall’s Theorem (and really using it). Let $X$ be a finite set and $\mathcal{S}$ a set of subsets of $X$, such that there are no distinct $S, T \in \mathcal{S}$ with $S \subset T$. Then

$$|\mathcal{S}| \leq \left\lfloor \frac{|X|}{|X|/2} \right\rfloor.$$  

*10. Deduce Hall’s Theorem from König’s Theorem, and deduce König’s Theorem from Hall’s Theorem. (For both theorems, your proof should really use the other theorem to obtain a relatively simple proof.)