1. Determine $R(P_3, P_3)$ and $R(K_{1,3}, K_3)$.
   - $R(P_3, P_3) = 5$: Let $K_5$ be 2-colored. Pick a vertex $x_0$. One color occurs at least twice at $x_0$, say it is red, and $x_0x_1, x_0x_2$ are red. If any of the edges from $x_1, x_2$ to the other two vertices $x_3, x_4$ is red, then we get a red $P_3$. Otherwise, for instance the path $x_1x_3x_2x_4$ is blue.
     For the lower bound, color $K_4$ with one triangle red, and the other three edges blue.
   - $R(K_{1,3}, K_3) = 7$: Let $K_7$ be colored red and blue. Pick a vertex $x_0$. If it has three red edges, we have a red $K_{1,3}$. Otherwise, $x_0$ has at least four blue edges $x_0x_1, x_0x_2, x_0x_3, x_0x_4$. If any edge between $x_1, x_2, x_3, x_4$ is blue, we get a blue $K_3$. Otherwise, $x_1, x_2, x_3, x_4$ form a red $K_4$, which contains a red $K_{1,3}$.
     To prove $R(K_{1,3}, K_3) > 6$, color $K_6$ so that the red edges form two red disjoint red triangles (so no red $K_{1,3}$), and the remaining edges are blue, which means they form a $K_{3,3}$ (so no blue $K_3$).

2. Show that any 2-coloring of the edges of $K_6$ contains at least two monochromatic triangles.
   There is one monochromatic triangle, say $x_1x_2x_3$ is red. Let $x_4, x_5, x_6$ be the other vertices. If the triangle $x_4x_5x_6$ is red, we are done, so we can assume that $x_4x_5$ is blue. If among the three edges from $x_4$ to $x_1, x_2, x_3$ there are two red edges, we would get a second red triangle, so we can assume that there are two blue edges from $x_4$ to $x_1, x_2, x_3$. Similarly, we can assume there are two blue edges from $x_5$ to $x_1, x_2, x_3$. Then there must be a blue edge from $x_4$ and a blue edge from $x_5$ that go the same vertex among $x_1, x_2, x_3$. This gives a blue triangle.

3. Show that there exists an $N$ such that if the integer box $\{(x, y) : 1 \leq x, y \leq N\}$ is 2-colored, then there is a monochromatic rectangle, i.e. a rectangle with all four corners the same color.
   We claim that $N = 5$ works. Suppose that we have a coloring of the $5 \times 5$ box without a monochromatic rectangle. There must be points $(x_1, 1), (x_2, 1)$, and $(x_3, 1)$ that all have the same color, say red.
   For each $i$, at most one other point of the form $(x_i, y)$ can be red. Indeed, suppose that for instance $(x_1, y_1)$ and $(x_1, y_2)$ are red, with $y_1, y_2 > 1$. Then each of the points $(x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)$ must be blue, or we’d get a red rectangle. But then these form a blue rectangle.
   Therefore, each of $(x_1, 1), (x_2, 1), (x_3, 1)$ has at least three blue points above it. Then there are two blue points above $(x_1, 1)$ which are at the same height as two blue points above $(x_2, 1)$, so they form a blue rectangle.
   For $N = 4$ it is not hard to construct a coloring without a monochromatic rectangle.

4. Prove that every 2-coloring of $K_n$ contains a monochromatic spanning tree.
   Pick a vertex $x$. If all edges of $x$ are red, they form a red spanning tree. Otherwise, $x$ has a red edge $xy$ and a blue edge $xz$. Apply induction to find a monochromatic spanning tree in $G - x$. If it is red, add $xy$, and if it is blue, add $xz$. 

5. Let \( T \) be a tree with \( t \) vertices. Prove that \( R(K_s, T) = (s - 1)(t - 1) + 1 \).

Color \( K_{(s-1)(t-1)+1} \) with red and blue. If every vertex has blue degree at least \( t - 1 \), then by a lemma from Lecture 10, there is a blue \( T \). Otherwise, there is a vertex \( x \) with blue degree at most \( t - 2 \), so red degree at least \( (s - 1)(t - 1) + 1 - (t - 2) > (s - 2)(t - 1) + 1 \). Let \( H \) be the 2-colored complete subgraph on the at least \( (s - 2)(t - 1) + 1 \) vertices that are connected to \( x \) by a red edge. By induction, \( H \) has a red \( K_{s-1} \) or a blue \( T \). The red \( K_{s-1} \) would give a red \( K_s \) in the whole graph.

*6. Determine \( R(K_3, K_{2,2}) \) and \( R(K_{2,2}, K_{2,2}) \).

*7. Let \( 2K_3 \) be the graph consisting of two disjoint triangles. Prove that \( R(2K_3, 2K_3) = 10 \).