1. Show that if $G$ is $K_{s,t}$-free, and $s \leq t$, then $|E(G)| \leq \frac{1}{2}(t-1)^{1/s}|V(G)|^{2-\frac{1}{s}} + \frac{1}{2}s|V(G)|$.

As in Lecture 11, we count stars $K_{1,s}$, which now gives

$$\sum_{x \in V(G)} \left( \frac{d(x)}{s} \right) \leq (t-1) \binom{n}{s} \leq (t-1) \frac{n^s}{s!}.$$  

Jensen’s inequality and rough estimates give

$$\sum_{x \in V(G)} \left( \frac{d(x)}{s} \right) \geq n \cdot \left( \frac{\sum d(x)/n}{s} \right) = n \cdot \left( \frac{2e/n}{s} \right) \geq n \cdot \left( \frac{2e/n - s}{s!} \right).$$

This gives

$$\left( \frac{2e/n - s}{s!} \right)^s \leq (t-1)n^{s-1},$$

which finally leads to

$$e \leq \frac{1}{2}(t-1)^{1/s}n^{2-\frac{1}{s}} + \frac{1}{2}sn.$$

2. (a) Let $P$ be a set of $n$ points in $\mathbb{R}^2$ and $C$ a set of $n$ circles. Show that the number of pairs $(p, c) \in P \times C$ such that $p \in c$ is at most $Cn^{5/3}$ for some constant $C$.

(b) Let $P$ be a set of $n$ points in $\mathbb{R}^2$. Show that the number of pairs of points from $P$ that have distance 1 is at most $Cn^{3/2}$ for some constant $C$.

(a): Let $G$ be the graph with vertex set $P \cup C$, with an edge between $p$ and $c$ if $p \in c$. There is no $K_{3,3}$ in this graph, because any three points are together contained in at most one circle (because any three non-collinear points determine a unique circle, and any three collinear points are on no circle), and any three circles intersect in at most two points (actually, any two circles already intersect in at most two points). Thus Problem 1 gives $|E(G)| \leq \frac{2^{1/3}}{2}|V(G)|^{5/3} + \frac{3}{2}|V(G)| \leq 3|V(G)|^{5/3}$.

(b) Let $G$ be the graph with vertex set $P$ and an edge between any two points that are at distance 1. Given any two points, there are at most two points that are at distance 1 from both (because these points are intersection points of the two unit circles around the two given points). Thus the graph has no $K_{2,3}$, so by applying Problem 1 we obtain $|E(G)| \leq \frac{\sqrt{2}}{2}|V(G)|^{3/2} + |V(G)| \leq 2|V(G)|^{3/2}$.

3. Let $P$ be a set of $n$ points in $\mathbb{R}^2$, such that no two points are more than distance 1 apart. Show that there are at most $n^2/3$ pairs of points whose distance is greater than $1/\sqrt{2}$.

Let $G$ be the graph with vertex set $P$, with an edge between any two points that are more than $1/\sqrt{2}$ apart. We claim that this graph contains no $K_4$. By Turán’s Theorem, this implies that $G$ has at most $n^2/3$ edges.

Given any four points, there must be three that form an angle of at least 90 degrees. In the triangle formed by those three points, the two sides adjacent to the angle cannot both be more than $1/\sqrt{2}$, because then the opposite side would have to be longer than 1. This shows the four points we started with cannot form a $K_4$. 


4. Let $H$ be a graph. A graph $G$ is called $H$-saturated if $G$ contains no $H$, but adding any edge to $G$ creates an $H$. We know that the maximum number of edges in a $K_3$-saturated graph is $\frac{1}{4}|V(G)|^2$. Determine the minimum number of edges in a $K_3$-saturated graph.

The minimum number is $|V(G)| - 1$, and the unique minimum $K_3$-saturated graphs are the stars. Stars are clearly $K_3$-saturated. Any graph with fewer edges is disconnected, which implies it is not $K_3$-saturated, since we can add an edge between components without creating a $K_3$. This proves that $|V(G)| - 1$ is the minimum.

To show that stars are the unique minimum $K_3$-saturated graphs (which the question does not ask for), observe that any other connected graph with the same number of edges is a tree. Any non-star tree contains a path with 3 edges, so connecting the endpoints of this path does not create a $K_3$ (it creates a $C_4$, and adding an edge to a tree creates only one cycle).

5. Recall that $P_k$ is the path with $k$ edges. Show that if a graph $G$ contains no $P_k$, then $|E(G)| \leq \frac{1}{2}(k-1)|V(G)|$. (Note that this improves by a factor $\frac{1}{2}$ the bound that we have seen for trees.)

In a bonus problem on the first problem set, we saw that any connected graph $G$ contains a path of length at least $2\delta(G)$ (unless $2\delta(G) > |G| - 1$). Therefore, if $\delta(G) > \frac{1}{2}(k-1)$, then $G$ contains a path of length $2\delta(G) > k - 1$, so it contains a path of length $k$.

We prove by induction on $|V(G)|$ that if $|E(G)| > \frac{1}{2}(k-1)|V(G)|$, then $G$ contains a $P_k$.

First suppose that $G$ is not connected. Then $G$ has a connected component $G_0$ such that $|E(G_0)| > \frac{1}{2}(k-1)|V(G_0)|$. Indeed, if no component satisfied this, then summing up the edges of the components would give a contradiction. Thus, by induction, $G_0$ contains a $P_k$, hence so does $G$.

Now suppose that $G$ is connected. Then by the first paragraph above, we are done if $\delta(G) > \frac{1}{2}(k-1)$, so we can assume that $G$ has a vertex $x$ of degree $d(x) \leq \frac{1}{2}(k-1)$.

Consider $G - x$. We have

$$|E(G - x)| = |E(G)| - d(x) > \frac{1}{2}(k-1)|V(G)| - \frac{1}{2}(k-1) = \frac{1}{2}(k-1)|V(G - x)|.$$

By induction, $G - x$ contains a $P_k$, hence so does $G$. 

*6. What is the maximum number of edges in a graph on \( n \) vertices that has precisely one triangle?

Let \( G \) be a graph with \( n \) vertices and exactly one triangle \( T \). Let \( H \) be the subgraph of \( G \) on \( V(G) \setminus V(T) \). It is triangle-free, so by Mantel’s Theorem \( |E(H)| \leq \left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor \). A vertex in \( H \) has at most one edge to \( T \), since otherwise we get a second triangle. Thus there are at most \( n-3 \) edges between \( H \) and \( T \), so we have

\[
|E(G)| \leq 3 + (n - 3) + |E(H)| = \left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor + n.
\]

We claim that this is the exact maximum. Start with the graph

\[ K_{\left\lfloor \frac{1}{4}(n-3) \right\rfloor, \left\lceil \frac{1}{2}(n-3) \right\rceil}, \]

which has \( n-3 \) vertices and \( \left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor \) edges. Add a vertex on the side with \( \left\lfloor \frac{1}{4}(n-3) \right\rfloor \) vertices and connect it to all \( \left\lceil \frac{1}{2}(n-3) \right\rceil \) vertices on the other side, then add a vertex on the side with \( \left\lceil \frac{1}{2}(n-3) \right\rceil \) vertices and connect it to all \( \left\lfloor \frac{1}{2}(n-3) \right\rfloor + 1 \) vertices on the other side. We have the bipartite graph \( K_{\left\lfloor \frac{1}{4}(n-1) \right\rfloor, \left\lfloor \frac{1}{2}(n-1) \right\rceil} \), which has no triangle, and from the way we obtained it we see that it has

\[
\left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor + \left\lfloor \frac{1}{2}(n-3) \right\rfloor + \left\lceil \frac{1}{2}(n-3) \right\rceil + 1 = \left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor + n - 2
\]

edges. We could also simply have observed that

\[
\left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor + n = \left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor + 2,
\]

but checking this directly is an annoying calculation. Finally, add one vertex and connected to one vertex on each side. This creates exactly one triangle, and the resulting graph has \( \left\lfloor \frac{1}{4}(n-3)^2 \right\rfloor + n \) edges, which proves that this is the maximum.

*7. What is the maximum number of edges in a \( C_5 \)-free graph on \( n \) vertices?

We have not yet found a solution to this problem, but the answer should be \( \left\lfloor \frac{1}{4}n^2 \right\rfloor \), just like for triangles.