1. **Show that if** \(|E(G)| > \frac{|V(G)|^2}{4}\), **then** \(G\) **contains** \(\square\), **unless** \(|V(G)| = 3\).

   By Mantel’s Theorem \(G\) contains a triangle \(T\), and if it contains no \(H = \square\), then this triangle is disconnected from \(G' = G - T\). But by induction \(G'\) contains an \(H\), since
   \[|E(G')| = |E(G)| - 3 > \frac{|V(G)|^2}{4} - 3 \geq \frac{(|V(G)| - 3)^2}{4},\]

   where the second inequality is true if \(3 \leq \frac{3}{2}|V(G)| - \frac{9}{4}\), or \(|V(G)| \geq \frac{7}{2}\).

   Since we took a step of size 3 in the induction, we need the base cases \(|V(G)| = 1, 2, 6\) (given that for \(|V(G)| = 3\) the claim is not true). The first two are trivial because the inequality is not possible. For \(|V(G)| = 6\), the condition is \(|E(G)| > 9\), so \(|E(G)| \geq 10\).

   Then there is a triangle, and if there is no \(H\), then the other three vertices have at most 3 other edges, contradiction.

2. **Recall from Lecture 4 that** \(\alpha(G) + vc(G) = |V(G)|\). **Prove that if** \(G\) **is triangle-free, then** \(|E(G)| \leq \alpha(G) \cdot vc(G)\), **and use this to reprove Mantel’s Theorem.**

   If \(G\) has no triangle, then for every vertex \(x \in V(G)\), the neighborhood \(N(x)\) is an independent set. Hence
   \[d(x) = |N(x)| \leq \alpha(G).\]

   Let \(S \subset V(G)\) be a minimum vertex cover, so that \(|S| = vc(G)\). Every edge has at least one endpoint in \(S\), so
   \[|E(G)| \leq \sum_{x \in S} d(x) \leq \sum_{x \in S} \alpha(G) \leq \alpha(G) \cdot vc(G).\]

   Then the basic inequality \(4ab \leq (a+b)^2\) (which follows from \((a+b)^2 - 4ab = (a-b)^2 \geq 0\)) gives
   \[|E(G)| \leq \alpha(G) \cdot vc(G) \leq \frac{\alpha(G) + vc(G)}{4} \leq \frac{|V(G)|^2}{4}.\]

3. **Let** \(G\) **be a graph with** \(n\) **vertices and** \(e\) **edges.** **Show that the number of triangles containing an edge** \(xy\) **is at least** \(d(x) + d(y) - n\). **Use this to prove that** \(G\) **has at least**
   \[\frac{4e}{3n} \left( e - \frac{n^2}{4} \right)\]

   **triangles. Deduce that if** \(e \geq \left(\frac{1}{4} + c\right)n^2\), **then** \(G\) **has at least** \(2c\binom{n}{3}\) **triangles.**

   The number of triangles based on an edge \(xy\) is
   \[|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \geq d(x) + d(y) - n.\]

   Hence the total number of triangles is at least
   \[\frac{1}{3} \sum_{xy \in E(G)} (d(x) + d(y) - n) = \frac{1}{3} \left( \sum_{x \in V(G)} d(x)^2 \right) - \frac{en}{3} \geq \frac{1}{3n} (2e)^2 - \frac{en}{3} = \frac{4e}{3n} \left( e - \frac{n^2}{4} \right).\]

   When \(e = \left(\frac{1}{4} + c\right)n^2\), the number of triangles is at least
   \[\frac{4\left(\frac{1}{4} + c\right)n^2}{3n} \cdot cn^2 = \frac{(1 + 4c)c}{3} n^3 \geq \frac{2c}{6} n^3 \geq 2c \binom{n}{3}.\]
4. Let $G$ have $n$ vertices and $e$ edges. Prove that if $G$ is $K_{2,2}$-free, then $e \leq \frac{1}{4}n(1 + \sqrt{4n - 3})$.

In the proof in the lecture we obtained that if $G$ is $K_{2,2}$-free, then

$$\binom{n}{2} \geq \sum_{x \in V(G)} \binom{d(x)}{2},$$

and we used rough bounds. We will now be more careful. Recall that by Cauchy-Schwarz

$$\sum_{x \in V(G)} \binom{d(x)}{2} = \frac{1}{2} \sum d(x)^2 - \frac{1}{2} \sum d(x) = \frac{1}{2n} \left( \sum d(x) \right)^2 - \frac{1}{2} \sum d(x) = \frac{2e^2}{n} - e.$$ 

Thus we have

$$\frac{1}{2}n^2 - \frac{1}{2}n \geq \frac{2e^2}{n} - e.$$ 

Multiplying by $2n$ and rearranging, we have

$$4e^2 - (2n) \cdot e + (n^2 - n^3) \leq 0.$$ 

By solving the quadratic equation, we get

$$e \leq \frac{1}{8} \left( 2n + \sqrt{(2n)^2 - 16(n^2 - n^3)} \right) = \frac{1}{4}n \left( 1 + \sqrt{4n - 3} \right).$$

5. Let $L$ be a set of $n$ lines in the plane and $P$ a set of $n$ points in the plane. Give an upper bound on the number of pairs $(p, \ell) \in P \times L$ with $p \in \ell$.

Define a bipartite graph $G$ on $L \cup P$ with an edge between $p$ and $\ell$ if $p$ lies on $\ell$. Then $G$ does not contain a $K_{2,2}$, since that would mean there are two distinct lines intersecting in two distinct points. Then the theorem from class gives that the number of incidences $p \in \ell$ is at most the number of edges in this graph, which is at most $(2n)^{3/2} = 2^{3/2}n^{3/2}$.

Comment: This is not best possible. One of the most famous theorems in combinatorial geometry, the Szemerédi-Trotter Theorem, gives the bound $cn^{4/3}$, and this is best possible, up to the exact value of $c$. 

Let $n$ be the number of vertices and $e$ the number of edges. We claim that the maximum number is $\lfloor n^2/4 \rfloor$, just like for triangles (with one exception, $n = 3$, when $e = 3$ is the maximum instead of $\lfloor 3^2/4 \rfloor = 2$). This would be sharp because $K_{\lfloor n^2/4 \rfloor,\lceil n^2/4 \rceil}$ contains no triangle, so also no $F = \square$.

Suppose $e > \lfloor n^2/4 \rfloor$, which implies $e \geq \frac{n^2}{4} + 1$, and there is no $F$. By Mantel’s Theorem there must be a triangle; let $T$ be a triangle, and let $S = G - T$ be the rest of the graph. Consider the edges between $T$ and $S$: there can be only $n - 3$ of these, since no 2 of these edges can meet at the same vertex of $S$, because that would give an $F$. So we have at most $n$ edges outside of $S$.

Now we can use induction: $S$ has no $F$ either, so $|E(S)| \leq \lfloor \frac{(n-3)^2}{4} \rfloor$, hence

$$e \leq n + e(S) \leq n + \frac{n^2}{4} - \frac{3}{2} n + \frac{9}{4} = \frac{n^2}{4} - \frac{n}{2} + \frac{9}{4}.$$ 

This is a contradiction to $e \geq \frac{n^2}{4} + 1$, unless $-\frac{n}{2} + \frac{9}{4} \geq 1$, or $n \leq \frac{10}{3}$, so $n \leq 2$.

Because our induction has “step size” 3, we’ll need 3 base cases.

For $n = 1, 2$ the claim is trivially true. But for $n = 3$, we can have $e = 3 > 3^2/4$, but no $F$, so in this case the claim is actually not true. We should exclude it, and we’ll need to check $n = 6$ to make the induction work.

For $n = 6$, $\lfloor 6^2/4 \rfloor = 9$, so suppose $e = 10$ and the graph contains no $F$. Then there must be a triangle, and 3 other vertices. There are at most 3 edges between the triangle and the 3 other vertices, and 3 edges between them, contradiction.