Questions

1. We toss a fair coin $n$ times. What is the expected number of ‘runs’? Runs are consecutive tosses with the same result. For instance, the toss sequence HHTTHTH has 5 runs.

2. For a permutation $\pi$, let $f(\pi)$ be the number of fixed points of $\pi$. What is $E[f(\pi)]$ for a random permutation $\pi$ on $n$ elements.

3. The number of left maxima for a permutation $\pi$ of $\{1, \ldots, n\}$ is defined to be the number of indices $i \in [n]$ such that $\pi(i) > \pi(j)$ for all $j < i$. Using linearity of expectation, compute the expected number of left maxima for a random permutation?

4. Let $X$ be a set of $n$ elements, and $\mathcal{M}$ a set system on $X$, i.e., $\mathcal{M} = \{S_1, \ldots, S_m\}$, where $S_i \subseteq X$ and $|S_i| = k$ for all $i = 1 \ldots m$. Prove that if $m < 2^{k-1}$, then $X$ can be two-colored (i.e., each element of $X$ can be colored either ‘red’ or ‘blue’) such that no set $S_i$ is monochromatic (a set $S$ is monochromatic if all the elements in $S$ have the same color).

5. Can you construct a tournament $T$ on 6 vertices such that for any pair of vertices $u, v \in T$, there is a third vertex $w$ such that $w$ beats both $u$ and $v$? What about a tournament with 7 vertices?

Bonus Problem. Prove that there exist four positive integers $a_1, a_2, a_3, a_4$ such that for any integer $w \in \{1, \ldots, 40\}$, there exist $c_i \in \{-1, 0, +1\}$, $i = 1, \ldots, 4$, such that $w = c_1 \cdot a_1 + c_2 \cdot a_2 + c_3 \cdot a_3 + c_4 \cdot a_4$.

10 points.