Questions

1. Given a set system $F$ over the base set $\{1, \ldots, n\}$, we call $F$ semi-independent if it contains no three sets $A, B, C$ such that $A \subset B \subset C$. Prove that $|F| \leq 2\binom{n}{\lfloor n/2 \rfloor}$.

2. Let $a_1, \ldots, a_n$ be real numbers with $|a_i| \geq 1$. Let $p(a_1, \ldots, a_n)$ be the number of vectors $(\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i = \pm 1$, such that $-1 < \sum_{i=1}^{n} \epsilon_i a_i < 1$.

   Prove that for any $a_1, \ldots, a_n$, we have $p(a_1, \ldots, a_n) \leq \binom{n}{\lfloor n/2 \rfloor}$.

3. Let $X$ be an $n$-element set, and let $S_1, \ldots, S_n$ be subsets of $X$ such that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$. Prove that at least one set has size at most $C\sqrt{n}$ for some absolute constant $C$.

4. Let $t(j)$ denote the number of divisors of the number $j$. Give an expression for the number $\sum_{i=1}^{n} t(j)$.

5. Given a set $P$ of $n$ points, and a set $L$ of $n$ lines in the plane, an incidence is a pair $(p, l)$, where $p \in P$, $l \in L$, and the point $p$ lies on the line $l$. Prove that given any set of $n$ distinct lines $L$ and $n$ distinct points $P$, the number of incidences are at most $3n^{1.5}$.

**Bonus Problem.** You are given a set $P$ of 10 integers from the set $\{1, \ldots, 100\}$. Prove that one can always find two disjoint subsets of $P$ such that the sum of the elements in the two sets are equal. For example, given the set $\{8, 15, 23, 59, 61, 70, 75, 88, 91, 97\}$, the two sets are $\{8, 15, 97\}$ and $\{59, 61\}$.

10 points.