Exam Guide – Advanced Discrete Mathematics 2013 – EPFL

The list below contains the theorems that you should be able to prove on the exam, and the problems that you should be able to solve. For theorems not on this list, you should only be familiar with the general idea and the relevant definitions; where below it says 'statement of...', it means you should know the statement, but not the proof. Problems that are not on the list you can ignore; for problems that had hints, we will not necessarily give those hints on the exam.

The exam will consist of 6 questions: 4 will be theorems or problems from below, and 2 will be "new" problems. We will try to keep a balance between the 4 different parts of the course, and between easier and harder questions. Note that understanding the general idea of a proof or solution is more important than the smaller details, and we will grade solutions accordingly.

Linear algebra

- L2: Generalized Fisher's inequality
- L2: Equiangular lines
- L2: Graham-Pollak Theorem
- L5: Oddtown Theorem
- L5: L-intersecting sets
- L6: Equilateral sets
- L6: Two-Distance Sets
- *Problems:* (1:3,4,6), (4:3,4,5)

Eigenvalues

- L8: Examples of spectra (except C_n)
- L8: $d_{\text{avg}} \leq \lambda_1 \leq d_{\max}$
- L8: Number of closed walks
- L9: Bipartiteness
- L9: $K_{10} \neq 3$ Pet
- L9: Windmill theorem
- *Problems:* (7:3,4), (8:2,3), (9:2,4)

Probability

- L3: Ramsey upper and 2 lower bounds
- L3: Large girth and chromatic number
- L4: Turán's Theorem
- L4: Statement of Kővari-Sós-Turán
- L7: Crossing Lemma
- L7: Szemerédi-Trotter Theorem
- L7: Sum-Product Theorem
- *Problems:* (2:2,3,4), (3:2,3,5), (6:4,5,6)

Regularity

- L11: Degree Lemma
- L11: Triangle Counting Lemma
- L11: Statement of Szemerédi Regularity Lemma, General Counting Lemma and General Removal Lemma
- L11: Triangle Removal Lemma
- L12: Roth's Theorem
- L12: Erdős-Stone Theorem
- *Problems:* (10:2,4)

Linear algebra

- L2: *Generalized Fisher's inequality:* A λ-intersecting ($\lambda \geq 1$) set system $(X, \mathcal{S})$ satisfies $|\mathcal{S}| \leq |X|$. This is tight and there are two very different tight examples (P1.3).
- L2: *Equiangular lines:* A set of equiangular lines in $\mathbb{R}^d$ has size $\leq \binom{d+1}{2}$.
- L2: *Graham-Pollak Theorem:* A complete graph K_n cannot be decomposed into $\leq n-2$ complete bipartite graphs.
- L5: *Oddtown Theorem:* A set system $(X, \mathcal{S})$ with odd sets and even intersections has $|\mathcal{S}| \leq |X|$.
- L5: *L-intersecting set systems:* If a set system $(X, \mathcal{S})$ has all $|S| \bmod p \notin L$ and $|S \cap T| \bmod p \in L$, then $|\mathcal{S}| \leq \sum_{i=0}^{|L|} \binom{X}{i}$
- L6: *Equilateral sets:* If $S \subset \mathbb{R}$ is equilateral, then $|S| \leq d+1$.
- L6: *Two-Distance Sets:* If $S \subset \mathbb{R}$ is a two-distance set, then $|S| \leq \binom{d}{2} + 3d + 2$.
- P1.4: If n points in $\mathbb{R}^2$ are not collinear, then there are at least n lines that pass through at least two of these points.
- P1.6: If $A_1, \dots, A_{n+1} \subset X$ and $|X| = n$, then there are two disjoint sets I, J of indices such that $\cup_{i \in I} A_i = \cup_{j \in J} A_j$.
- P4.4: If $A \subset \mathbb{F}_3^n$ has the property that for all distinct vectors $a, b \in A$, there is a coordinate $i \in [n]$ such that $a_i - b_i \equiv 1 \bmod 3$, then $|A| \leq 2^n$.
- P4.5: The graph G defined by $V(G) = [m]^{(3)}$, $E(G) = \{ST : |S \cap T| = 1\}$ contains no K_k or $\overline{K_k}$.

Probability

- L3: *Ramsey bounds:* The Ramsey numbers $R(k)$ exist and $R(k) \leq \binom{2k-2}{k-1}$. We have $R(k) > 2^{k/2-1}$ and the improvement $R(k) > (1 - o(1))(k/e)2^{k/2}$.
- L3: *Large girth and chromatic number:* For any k, l there is a graph G with $g(G) > k$ and $\chi(G) > l$.
- L4: *Turán's Theorem:* If a graph contains no K_t, then $e \leq \frac{1}{2}(1 - \frac{1}{t-1})n^2$.
- L4: *Statement of Kővari-Sós-Turán:* For $s \leq t$ there is a constant c such that if a graph contains no K_{st}, then $e \leq cn^{2-1/s}$.
- L7: *Crossing Lemma:* If G is a graph with $e \geq 4n$, then $\mathrm{cr}(G) \geq \frac{e^3}{64n^2}$.
- L7: *Szemerédi-Trotter Theorem:* Let P be a set of n points in $\mathbb{R}^2$ and L a set of m distinct lines. Then the number of incidences between P and L is $I(P, L) \leq 4(mn)^{2/3} + m + 4n$.
- L7: *Sum-Product Theorem:* There is a $c > 0$ such that $\forall A \subset \mathbb{R}_{\geq 0}$ we have $\max\{|A+A|, |A \cdot A|\} \geq c|A|^{5/4}$.
- P2.2: The minimum size of a non-two-colorable family of n-subsets of a finite set is $m(n) \geq 2^{n-1}$.
- P2.3: If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices with the following property: for any set X of k vertices there is a vertex $v \in X$ such that all the k edges connecting it with X are directed towards v.

- P2.4: If G has m edges and a matching of size μ, then it has a bipartite subgraph with at least $(m+\mu)/2$ edges.
- P3.2: For all n there exists a $K_{r,r}$-free graph with n vertices and at least $c_r n^{\frac{2r}{r+1}}$ edges.
- P3.3: If we have n points inside a unit disc in the plane, then there are $\geq (n^2 - 3n)/6$ pairs of points at distance not ≤ 2.
- P3.5: There is a constant c such that if X is an n-element set and $S_1, \ldots, S_n$ are subsets of X such that $|S_i \cap S_j| \leq 1$ whenever $i = j$, then at least one of the S_i has size $\leq cn$.
- P6.5: There is a constant c such that the number of incidences between n points and m unit circles in the plane is at most $c(m^{2/3}n^{2/3} + m + n)$.
- P6.6: There is a constant c with the property that for any set n points in the plane, the number of pairs of the points at distance exactly 1 is at most $cn^{4/3}$.

Eigenvalues

- L8: $\text{Spec}(K_n) = (n-1)^1(-1)^{n-1}$, $\text{Spec}(K_{mn}) = (\sqrt{mn})^1(0)^{m+n-2}(-\sqrt{mn})^1$, $\text{Spec}(Pet) = (3)^1(1)^5(-2)^4$
- L8: The largest eigenvalue λ_1 of a graph satisfies $d_{\text{avg}} \leq \lambda_1 \leq d_{\text{max}}$.
- L8: The number of closed walks of length k in G equals $\sum \lambda_i^k$.
- L9: A graph is bipartite if and only if its spectrum is symmetric.
- L9: K_{10} cannot be decomposed into 3 Petersen graphs.
- L9: *Windmill theorem:* If G has the property that every two vertices have exactly one common neighbor, then it is a windmill graph.
- P7.3: If G is d-regular, then the multiplicity of λ_1 equals the number of connected components of G.
- P7.4: If G is connected, then its diameter is strictly less than its number of distinct eigenvalues.
- P8.2: There are non-isomorphic graphs that have the same spectrum, but can be connected or disconnected, so connectedness cannot always be determined from the spectrum.
- P8.3: It can be proved with eigenvalues that if a graph has $> n^2/4$ edges, then it contains a K_3.
- P9.2: The largest eigenvalue λ_1 equals the maximum value of $x^T A_G x$ over all $x \in \mathbb{R}^{|V(G)|}$ with $x^T x = 1$.

Regularity

- L11: *Degree Lemma:* Consider an ϵ-regular bipartite graph (A, B) with density $d(A, B) > \epsilon$. The number of vertices in A with fewer than $(d(A, B) - \epsilon)|B|$ neighbors in B is less than $\epsilon|A|$. Similarly, the number of vertices in A with more than $(d(A, B) + \epsilon)|B|$ neighbors in B is less than $\epsilon|A|$.

- L11: *Triangle Counting Lemma:* Consider a tripartite graph $G = (A, B, C)$ such that (A, B), (A, C) and (B, C) are ϵ-regular with densities at least 2ϵ. The number of triangles in G is at least

$$(1 - 2\epsilon)(d(A, B) - \epsilon)(d(A, C) - \epsilon)(d(B, C) - \epsilon)|A||B||C|.$$

- L11: *Statement of Szemerédi Regularity Lemma:* For every $\epsilon > 0$ there is $M \in \mathbb{N}$ such that every graph G with at least $1/\epsilon$ vertices admits a partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$ such that
  - $1/\epsilon \leq k \leq M$,
  - $|V_0| \leq \epsilon v(G)$,
  - $|V_1| = \ldots = |V_k|$,
  - all but at most ϵk^2 pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ϵ-regular.

- L11: *Triangle Removal Lemma:* For every $\alpha > 0$ there is $\delta > 0$ such that every graph G that cannot be made triangle-free by removing fewer than αn^2 edges contains at least δn^3 triangles, where $n = v(G)$.

- L12: *Roth's Theorem:* For every $\beta > 0$ there is $n \in \mathbb{N}$ such that every set $S \subseteq \{1, \ldots, n\}$ with $|S| \geq \beta n$ contains a 3-term arithmetic progression.

- L12: *Statement of General Counting Lemma:* For any $d \in (0, 1]$ and any graph H, there are $\epsilon \in (0, d/2)$ and $\ell_0 \in \mathbb{N}$ such that the following holds: if $G = (V_1, \ldots, V_k)$ is a k-partite graph with $|V_1| = \ldots = |V_k| = \ell \geq \ell_0$ and $\sigma \colon V(H) \to \{1, \ldots, k\}$ is a mapping such that $(V_{\sigma(i)}, V_{\sigma(j)})$ is ϵ-regular with density at least d whenever $ij \in E(H)$, then G contains at least one and asymptotically $\Omega(\ell^{v(H)})$ copies of H.

- L12: *Statement of General Removal Lemma:* For every graph H and every $\alpha > 0$ there is $\delta > 0$ such that every graph G with n vertices that cannot be made H-free by removing fewer than αn^2 edges contains at least $\delta n^{v(H)}$ copies of H.

- L12: *Erdős-Stone Theorem:* For every graph H and every $\alpha > 0$ there is $n_0 \in \mathbb{N}$ such that every graph G with $n \geq n_0$ vertices and at least

$$\left(\frac{\chi(H) - 2}{\chi(H) - 1} \cdot \frac{1}{2} + \alpha\right) n^2$$

  edges contains a copy of H.

- P10.2: Let (A, B) be an ε-regular bipartite graph, with $A' \subset A$ and $B' \subset B$ such that $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$, where $\alpha > \varepsilon$. Then (A', B') is ε'-regular with $\varepsilon' = \max\{2\varepsilon, \varepsilon/\alpha\}$.

- P10.4: For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $A \subset [n]^2$ with $|A| \geq \varepsilon n^2$ contains three points of the form (x, y), $(x + a, y)$, and $(x, y + a)$ with $a \neq 0$.