Combinatorial Counting

- # subsets of an $n$-element set is $2^n$. At each element one has two choices, choosing it or not: 1, 2, 3, ...
- # subsets of an $n$-element set with an $n$-element set with an even number of elements is $2^{n-1} = \frac{1}{2} 2^n \iff$ Probability that a random subset of \{1, ..., n\} has an even number of elements is $\frac{1}{2}$.

Proof: $\forall A \subseteq \{2, ..., n-1\}$

\[
\begin{align*}
A & \cup \{1\} \\
\text{odd} & \iff \text{even}
\end{align*}
\]

- What is the probability that $A \subseteq B$ or $B \subseteq A$, when $A, B$ are randomly and independently chosen subsets of \{1, ..., n\}?

Proof: We need to count # pairs $(A, B)$ s.t. $A \subseteq B$ or $B \subseteq A$.

Let $A$ be such that $|A| = k$, then there are $2^{n-k}$ sets $B$ that contain $A$, and there are $2^k B$ s.t. $B \subseteq A$.

But there is one double counting so # pairs $(A, B)$

\[
A_k = 2^k + 2^{n-k} - 1.
\]

So all different possibilities need to be summed up: $N = \sum_{k=0}^{n} \binom{n}{k} (2^k + 2^{n-k} - 1)$.

So the sought for probability is $\rho = \frac{N}{2^n 2^n}$. An alternative way to derive $N$ is the following.
(n + x)(n+k) = \sum_{k=0}^{n} \binom{n}{k} x^k


Explain problems from problem sheets and proofs/derivations from lectures.

a) purely combinatorial proof.

So we proved by double counting that

\[
\sum_{k=0}^{n} \binom{n}{k} (k+2)^{n+2} = n
\]

Similarly, we proved by double counting that

\[
\sum_{k=0}^{n} \binom{n}{k} (k+2)^{n+2} = n
\]

Thus

\[
p = \frac{2n-2}{2n-1}
\]
Corollary: Set \( x = 1 \), then
\[
2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}
\]
Proof: \( 2^n = \# \) subsets = \( \# 0 \text{ elem. subset} + \# 1 \text{ elem. subset} + \ldots \)

Corollary: \( x = -1 \): \( 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \)

Adding the line for \( x = 1 \), and \( x = -1 \) gives:
\[
2^n = 2 \left( \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots \right), \quad \text{another proof of the statement on the first page, that } \# \text{ even element subsets} = \frac{2^n}{2}
\]

Proposition:
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k 2^{n-k-1} = 2 \cdot 3^n - 2^n
\]
Proof: By the binomial theorem, \( \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n \). Also,
\[
2^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right)^k = \left(1 + \frac{1}{2}\right)^n 2^n = 3^n
\]
Finally,
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad \square \quad (\text{Algebraic proof})
\]

What is the \# different ways one can distribute \( k \) identical coins among \( n \) people (restricted case: everyone can get 0 or 1 coin) \( k \leq n \Rightarrow \binom{n+k-1}{k} \).

Prop. \( \binom{n+k-1}{k} \)

Proof: \( \ldots \) coins, sequence of length \( n+k-1 \), with separators: \( \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( n-1 \) separators, \( k \) coins \( \square \)
ways one can write \( k \) as the sum of \( n \) non-negative integers = \( \binom{n+k-1}{k} \), i.e. \( k = x_1 + x_2 + \ldots + x_n \), \( x_i \geq 0 \), integers.

Estimating \( n! \):

\[
n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ for big } n. \text{ Necessary:}
\]

\[
\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1, \text{ (Stirling). Let's consider other estimates.}
\]

\[
n! = 1 \cdot 2 \cdot 3 \ldots n < n^{n-1}, \text{ assume } n = \text{even.}
\]

\[
\begin{array}{c|c|c|c}
1 & \frac{n}{2} & \ldots & n \\
\frac{n}{2} & \frac{n}{2} & \ldots & n \\
\vdots & \vdots & \ddots & \vdots \\
n & \frac{n}{2} & \ldots & \frac{n}{2}
\end{array}
\]

\[
n! \geq 1 \cdot \frac{n}{2} \cdot \ldots \cdot \frac{n}{2} \cdot \left(\frac{n}{2} + 1\right) \ldots n
\]

\[
\geq \left(\frac{n}{2}\right)^{n/2} \cdot n^{n/2} \geq \frac{n^n}{(\sqrt{2\pi})^n}, \text{ a non-trivial upper estimate.}
\]
Estimates for $n!$: 

$$\left(\frac{n}{e}\right)^{n} \leq n! \leq \left(\frac{n}{12}\right)^{n}$$

Stirling: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^{n}$

which means 

$$\lim_{n \to \infty} \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^{n}}{n!} = 1.$$ 

Thus: 

$$e \left(\frac{n}{e}\right)^{n} \leq n! \leq en \left(\frac{n}{e}\right)^{n}$$

Proof: Let us first estimate the natural logarithm of $n!$,

$$\log n! = \log 1 + \log 2 + \ldots + \log n$$

$$= \ln 1 + \int_{1}^{n+1} \ln x \, dx = x \ln x - x \bigg|_{1}^{n+1}$$

$$= (n+1) \ln (n+1) - (n+1) + 1 = (n+1) \ln (n+1) - n$$

How big is the error?

Look at the strip from zero to one and shift the error "triangle" into the strip:

An alternative is to consider an upper bound of the error by looking at rectangles as sketched above.
Thus \( n! \leq e^{(n+1)\ln(n+1)-n} = \frac{(n+1)^{n+1}}{e^n} \quad \forall n \in \mathbb{N} \)

\[
(n-1)! \leq \frac{n^n}{e^{n-1}}
\]

\[
\Rightarrow n! = n \cdot (n-1)! \leq ne\left(\frac{n^n}{e}\right).
\]

**Note:** "Integral method": \( f(1) + f(2) + \ldots + f(n) \leq \int f(x)\,dx \)

whenever \( f(x) \) is monotonically increasing.

Thus:

\[
\binom{n}{k} \leq \left(\frac{n}{e}\right)^k
\]

**Mnemonic:**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!}
\]

Proof:

\[
\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} \leq \left(\frac{en}{k}\right)^k
\]

\[
\leq \frac{n^k}{(k/e)^k} = \left(\frac{en}{k}\right)^k
\]

Include an \( e \) from first line, to make this proof correct.

Recall the binomial theorem,

\[
\binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{k}x^k \leq (1+x)^n, \quad 0 < x < 1
\]

Dividing both sides by \( x^k \) gives:

\[
\binom{n}{0}x^{-k} + \binom{n}{1}x^{1-k} + \ldots + \binom{n}{k}x^{k-k} \leq \frac{(1+x)^n}{x^k}
\]

Each coefficient \( x^{1-k} \) is larger than one,

\[
\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} \leq \frac{(1+x)^n}{x^k} \quad \forall 0 < x < 1
\]
Let $x = \frac{k}{n} < 1$ as $k < n$, then
\[
\frac{k}{n} \sum_{i=0}^{n} \left(\frac{k}{n}\right)^i \leq \left(1 + \frac{k}{n}\right)^n \leq \left(\frac{e^k}{n}\right)\left(\frac{k}{n}\right)^n
\]

Since
\[
1 + x \leq e^x \quad \forall x > 0,
\]
i.e. $e^x$ is convex, i.e.
always above its tangent line.

Here are another methods for estimating sequences

**Generating Function Method**

\[
a_0 + a_1 + \ldots + a_k \quad \xrightarrow{\text{map}} \quad a_0 + a_1x + a_2x^2 + \ldots + a_kx^k \quad \text{Taking derivatives gives coefficient.}
\]

**Inclusion/Exclusion Formula** (Principle of I/E $\rightarrow$ PIE)

\[
\left| A_1 \cup A_2 \right| = \left| A_1 \right| + \left| A_2 \right| - \left| A_1 \cap A_2 \right|
\]
\[
\left| A_1 \cup A_2 \cup A_3 \right| = \left| A_1 \right| + \left| A_2 \right| + \left| A_3 \right| - \left| A_1 \cap A_2 \right| - \left| A_1 \cap A_3 \right| - \left| A_2 \cap A_3 \right| + \left| A_1 \cap A_2 \cap A_3 \right|
\]

So there are 8 regions
an element can be.

Only $A_i$, $A_i \cap A_j$, $A_i \cap A_j \cap A_k$, one in none. So we can check the above formula by comparing the regions covered on each side.
For any $k$:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{\{i,j\} \in \binom{\{1,\ldots,n\}}{2}} |A_i \cap A_j|$$

$$+ \sum_{\{i,j,k\} \in \binom{\{1,\ldots,n\}}{3}} |A_i \cap A_j \cap A_k|$$

$$- \sum_{\{i,j,k,l\} \in \binom{\{1,\ldots,n\}}{4}} (-1)^{n+1} |A_i \cap A_j \cap A_k \cap A_l|$$

**Proof:** Every element that belongs to the union of the sets of the LHS can be counted only once on the RHS.

Pick an element that belongs to precisely $t$ of the sets $A_i$, $i \in \{1,\ldots,n\}$, $t \in [1, n]$. It was counted once on LHS.

$$1 = t - \binom{t}{2} + \binom{t}{3} - \binom{t}{4} + \ldots + (-1)^t \binom{t}{n}$$

which holds by the binomial formula:

$$1-t + \binom{t}{2} - \binom{t}{3} + \ldots = (1-1)^t = 0.$$

What is the probability that if we put $n$ letters randomly into $n$ envelopes then at least one letter is put in the right envelope? $\approx \frac{1}{e}$
Proof: Envelope 1 2 3 ... n \ n! \text{ possibilities.}

letters 5 10 3 ... 2

Let \( A_3 \) = \{ permutations \ where \ the \ third \ letter \ gets \ into \ the \ third \ envelope \}.

Then \( |A_1 \cup A_2 \cup \ldots \cup A_n| \) is \( \neq \) permutation where at least one letter ends up at the right place.

\[
\begin{align*}
|A_i| & = n(n-1)! - \left( \frac{n}{2} \right)(n-2)! + \left( \frac{n}{3} \right)(n-3)! - \ldots \\
& = n(n-1)! - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \ldots = n! \sum_{k=1}^{n} \frac{(-1)^k}{k!} \\
& \quad \xrightarrow{n \to \infty} \frac{1}{e} \quad \text{Taylor: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\end{align*}
\]

Lemma: Handshake Lemma

The number of people who shook an odd number of hands is even.

Proof: \[ \# \text{handshakes} = \deg(v), \quad \sum_{v \in V(G)} \deg(v) = 2 |E(G)|. \]

This was an example for a parity argument.
Extra bonus problem:

Let there be people living on a n x n grid, some of them infected. Each person has at most eight neighbors.

Rule 1: If ≥2 neighbors of a square are infected then the square gets infected.

Rule 2: Infection is eternal.

1. What is smallest # people that one can infect all squares?
2. Same game but neighbor is defined differently. Every square has at most 4 neighbors.

Lemma: (Sperner)

Having vertices on edges of big triangle and inside it, connect them manually such that no vertex pair crosses each other.

- The vertices are coloured in three different colours, (2,3 respectively).
- On the side opposite to 1, only vertices with colour 2 and 3 are allowed.
- But all the other vertices can be coloured arbitrarily.

Sperner's lemma says, then \exists a small triangle which has all three colours.

Proof: Put a vertex into each face of the small triangles and one outside the big triangle and connect two of those new vertices iff they are neighbors (they share a side) and they share a this side is of type ①——②.
This defines a new graph, say $G$. For the sake of contradiction, suppose $\exists$ a small three coloured triangle. Then $d(v)$ is even for every vertex $v$, except for $v_0$,

$$d(v) = 2 \quad (\star)$$

and $d(v_0)$ is odd. This can be seen when walking along an outer edge then there must be an odd number of colour changes. When there is a colour change $1(\rightarrow)2$ then there is an edge connecting to the outside $v_0$.

Now $\sum d(v) = \text{even} \implies \exists$ internal vertex with odd degree, contradicting $(\star)$. \hfill \Box

Remark: One could generalise this lemma to higher dimensions. Consider for example a tetrahedron, where one can also find a small internal tetrahedron, having four co"ut"coloured vertices.

Thus: (Brouwer's fixed point theorem)

Let $f: [0,1] \rightarrow [0,1]$ be a continuous function.

Then $\exists x_0 \in [0,1]$ s.t. $f(x_0) = x_0$

Proof: If $f(0) \neq 0$ then $f(0) > 1$, if $f(1) \neq 1$ then $f(1) < 1$.

and let $g(x) = f(x) - x$ then we get for the two cases:

$g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$

$\implies \exists x_0$ s.t. $g(x_0) = 0$ by Rolle. \hfill \Box
In higher dimensions, the theorem is more difficult:

Thus: let $f$ be a continuous mapping from one triangle to another, then $f(p) = p$ for some $p$.

(The shape does actually not matter so the region only need to be simply connected.)

Def.: Carycentric coordinates in the plane.
Assign to each point three coordinates,
$$p = \left( x(p), \lambda(p), y(p) \right)$$ such that $x + \lambda + y = 1$.

Fix three points of the plane and allocate a fraction of the unit mass to the three points.

So we define:
$$x(p) = \alpha \cdot x(p_1) + \lambda \cdot x(p_2) + y \cdot x(p_3)$$
$$y(p) = \alpha \cdot y(p_1) + \lambda \cdot y(p_2) + y \cdot y(p_3)$$

$p$ is the centre of gravity of the triangle.

So every point $p$ inside the triangle defined by $p_1, p_2, p_3$ has 3 coordinates $x(p), \lambda(p), y(p)$ such that the center of gravity of $p_1, p_2, p_3$ with these weights is $p$. E.g. when I can choose $\alpha, \lambda, y$ such that $p$ is in the center of gravity of $p_1, p_2, p_3$. 
Proof: (Brouwer's fixed-point theorem in 2D)

Define a coloring of all \( p \in \Delta \) in the following way:

- \( p \) has color 1 if \( \alpha(f(p)) < \alpha(p) \)
- \( p \) has color 2 if \( \beta(f(p)) < \beta(p) \)
- \( p \) has color 3 if \( \gamma(f(p)) < \gamma(p) \)

(Each point gets at least one color)

If \( f \) has no fixed point, i.e. \( f(p) \neq p \)

Clearly, \( P_1 = (x=1, y=0, z=0) \)

and Sperner's lemma applies, since on an edge there cannot be a vertex with color of the opposing big triangle vertex.

Consider the same triangle with a finer mesh, apply Sperner's lemma again and maybe the tri-color triangle is somewhere else. When making the mesh finer and finer then there is a point of accumulation of the small multicolored triangles. Call this point \( p \). This is the limit of a sequence of points colored in 1, 2, 3, respectively.
Since a sequence \( p_1^i, p_2^i, p_3^i, \ldots \) s.t. \( \alpha(p_1^i) > \alpha(f(p_1^i)) \)

\[ \Rightarrow \alpha(f(p)) = \alpha(p) . \]

And also \( \beta(f(p_2^i)) < \beta(p_2^i) \) and \( \gamma(f(p_3^i)) < \gamma(p_3^i) \) for corresponding sequences.

But \( \alpha(f(p)) + \beta(f(p)) + \gamma(f(p)) = 1 \) and also

\[ \alpha(p) + \beta(p) + \gamma(p) = 1 \]

which implies that \( f(p) = p \). \( \square \)

**Remark:** There are equivalent versions of Brouwer's fixed point theorem. So when squeezing a spherical shell into the plane, one pair of antipodal points will be mapped together.

\[ \mathbb{S}^2 \rightarrow \mathbb{R}^2, \quad f(p) = \{ f(p^x) \} . \]

**Theorem:** (Hedgehog theorem) One cannot comb a hedgehog.

Assumptions: Continuity and no hair sticks perpendicularly out from the surface.

**Remark:** Two-dimensional hedgehogs can be combed.
Recap exercise: \((\text{odd, even})\)

\[\mathbb{Z}^2 : (-1, 2), (2, -1), (0, 0)\]

Show that no matter how we select 5 points, there will be 2 among them such that the midpoint of the line joining them is also an integer point.

Proof: This can be solved via parity and pigeon hole principle.

Pigeon hole implies, given 5 five points, at least two must have the same parity, for example \((a, b), (a', b')\) s.t. \(a \equiv a' \mod 2\) and \(b \equiv b' \mod 2\). Hence their midpoint \(\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)\) must be \(\in \mathbb{Z}^2\).

Let us consider a "fractional" version of the pigeon hole principle. Given \(n\) glasses and 1\(L\) of liquid, there is at least one glass with \(\geq \frac{1}{n}\) liter and one with \(\leq \frac{1}{n}\) liter.

\(n\) glasses

E.g.: Consider an \(m \times k\) matrix \((a_{ij})\):

\[
\begin{array}{cccc}
    a_{11} & a_{12} & \cdots & a_{1k} \\
    a_{21} & \; & \; & \; \\
    \; & \; & \; & \; \\
    a_{m1} & \cdots & a_{mk} \\
\end{array}
\]

Assume the average of each row is at least one hundred, then there must be a column with average at least 100.
\[
\sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \ldots + \sum_{i=1}^{m} a_{ik} \geq 100mk
\]

By contradiction, if one column with sum \( \geq 100m \).

This was another example of double counting, since

\[
\sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij} = \sum_{j=1}^{k} \sum_{i=1}^{m} a_{ij}.
\]

**Thm.** (Extremal set theorem)

**Extremal Set Theory**

**Thm.** (Erdős-Ko-Rado theorem)

Let there be \( n \) elements, \( S = \{1, \ldots, n\} \), and \( k \)-element subsets. What is the maximal \# distinct \( k \)-subsets?

**E.g.** For \( k = 2 \), how many edges can we maximally pick among \( n \) points such that any ed two points has one in common.

point.
Theorem (Erdős-Ko-Rado): Intersecting

If \( n \geq 2k \) and \( F \) is an system of \( k \)-element subsets of \( X \) where \( |X| = n \), then

\[
|F| \leq \binom{n-1}{k-1}.
\]

\( F \) is called an intersecting system, i.e. \( F \cap F' \neq \emptyset \) for all \( F, F' \in F \).

Proof: By double counting. Assume \( F \) is intersecting, containing \( k \) elements each subset of \( X = \{1, \ldots, n\} \), i.e. \( \forall F \in F, |F| = k \).

How many ways are there to place \( n \) numbers on a circle?

\[
\# \text{ seatings} = (n-1)!
\]

\( F \in F \) shall be said to be consecutive in a seating \( s \), if the elements of \( F \) are sitting next to each other, i.e. form an interval.

We double count the number of pairs \( (s, F) \) where \( s \) a seating and \( F \) a consecutive element of \( F \) with respect to the seating \( s \).

\[
\#(s, F) = \sum_{F \in F} \left( \# \text{ seatings in which } F \text{ is consecutive} \right)
\]
Given a seating:

We can permute these neighbors.

\[ \binom{|F|}{k} \times k! \times (n-k)! \]

\[ \leq \binom{n-k}{k} \times (n-k)! \]

\[ \leq (n-1)! \times k \]

Hence \[ |F| \leq k! \times (n-k)! \leq (n-1)! \times k \]

\[ |F| \leq \frac{(n-1)! \times k}{(n-k)! \times k!} = \binom{n-1}{k-1} \]

\[ \square \]

**Def.:** A tree is a simple, connected acyclic graph.

**Prop.:** Any finite tree on \( n \) vertices has \( n-1 \) edges.

**Proof:** By induction on the number of vertices.

\[ n=1 \]

\[ n=2 \]

Claim: If \( n \geq 2 \) then every tree with \( n \) vertices has a vertex of degree 1 (a leaf). The proof is algorithmic.

Whenever you go to a new vertex it cannot be an already visited one as a tree is acyclic.
Suppose $T$ has $n$ vertices and the proposition is true for all trees with at most $n-1$ vertices. By claim, find a leaf $v$ in the above algorithm and delete it. So it remains to show that $T - \text{leaf } v$ is a tree. This holds clearly. □

Maxwell studied graphs for electrical networks, and inspired the following theorem:

**Theorem:** (Cayley's Theorem)

What is the number of different trees that one can define on $n \geq 2$ labeled vertices? $f(n) = n^{n-2}$.

E.g. $\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{array} \neq \begin{array}{c}
\begin{array}{c}
3 \\
2 \\
1
\end{array}
\end{array}$. (Two graphs are the same if for each vertex the list of the neighbors is the same).

**Proof:** For $n = 3$, $f(3) = 3$. \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{array} \Rightarrow \begin{array}{c}
\begin{array}{c}
\text{four of these}
\end{array}
\end{array}

$n = 4$ it is already harder: \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{array}
\Rightarrow (4) \cdot 2 = 12

$f(4) = 16$.

It is to show that $f(n) = n^{n-2}$.

(Prüfer's proof)

Assign a code word to each tree which has length $n - 2$ and its letters will be numbers between 1 and $n$ and
we can repeat the same number. Clearly \( \# \text{trees} \leq \# \text{code words} \).

How to see the one-to-one correspondence between the codes and the trees?

Consider the following algorithm to form a code for a given tree:

1. Pick the leaf with the smallest label. This leaf has a unique neighbor and write down its label. Next delete the leaf and recurse to get a code word. Applied to our example: \( 44929460 \).
2. Stop when only two vertices left.

To show 1-1 correspondence we need to show that the code is enough to reconstruct the tree in a unique way, i.e., no two trees have the same code.

**Proposition:** Let \( d_i \) = degree of vertex \( i \) in \( T \), then

\[
    d_i = (\# \text{occurrences of } i \text{ in the code}) + 1.
\]

For example:

- \( d_1 = 3 \), \( d_2 = 3 \), \( d_3 = 1 \), \( d_4 = 4 \), \( d_5 = 1 \), \( d_6 = 2 \)
- \( d_7 = 1 \), \( d_8 = 1 \), \( d_9 = 1 \), \( d_{10} = 1 \), for code \( 94429460 \).

As a check, \( \sum d(v) = 2E = 2(n-1) = 18 \) for \( n = 10 \) and a 10-node tree.

Proof of proposition: ?
\[ \binom{n}{\frac{n}{2}} \binom{n}{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} \quad \text{for odd } n? \]

**Thm:** (Sperner's theorem)

One can pick at most \( \binom{n}{\frac{n}{2}} \) subsets from \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) such that no subset is contained in another one.

E.g.: \( n=3 \), \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \), then are 8 subsets. What's the maximal \# of subsets such that no set is completely contained in another. Here it is 3, \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \), for example.

**Proof 1:** Inuitive idea: Split all subsets into chains (partition) into \( t \) chains. As they are disjoint \( t \) must be equal to the maximal number of non-pairwise contained sets.

\[ 2^n \lesssim A_1 \subseteq A_2 \subseteq \ldots \]

\[ B_1 \subseteq B_2 \subseteq \ldots \]

\( A_1 \subseteq A_2 \subseteq A_3 \ldots \) is a chain.

To find this partition is difficult. So we better count more chains, non-disjoint, and divide them then by to add doubly counted sets.

\( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n \). Pick some permutation order and construct from it a chain in the following way: E.g.

\( 4, 2, 3, 1, 4 \rightarrow \mathcal{E}_2, \mathcal{E}_5, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_5, \mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_5, \mathcal{E}_4, \mathcal{E}_2, \mathcal{E}_5, \mathcal{E}_4 \).

So we have \( n! \) permutations and each chain has \( n \) elements. A fixed set \( S \) of \( k \) elements is contained in \( k!(n-k)! \) chains.
let $M = \{ M_1, \ldots, M_k \}$, a set of sets where no set is a subset of another. We double count the following pairs. Let $M' \in M$ and $s$ be a permutation s.t. $M'$ was contained in the chain generated from permutation $s$. (For example with $n=3$ it is clear).

$\# \text{ such pairs} \leq n!$

Note that each permutation appears only once in each $(M, s)$, i.e. $(M, s), (M', s)$ is impossible, as $M \not\subset M' \wedge M' \not\subset M$. Let $|M'| = k$, then $M'$ appears in exactly $k! (n-k)!$ pairs. Thus

$$\sum_{i=1}^{k} \frac{1}{m_i! (n-|M_i|)!} = \# \text{ pairs } (M', s) \leq n!$$

Thus

$$\sum_{i=1}^{k} \frac{1}{|M_i|! (n-|M_i|)!} \leq 1.$$

Look at the binomial terms to see:

$$\sum_{i=1}^{k} \frac{1}{{n \choose 2}} \leq \sum_{i=1}^{k} \frac{1}{{m_i \choose 1}}$$

$$\frac{k}{{n \choose 2}} \leq 1 \implies k \leq \binom{n}{2}.$$
Proof 2: By induction.

To show: given any \(2^n\) subsets of \(E_1, \ldots, E_n\), one can partition them into \(\left(\frac{n}{2}\right)\) chains.

These chains shall be symmetric, e.g., \(\{1, 2, 3\}\) has subsets of size 0, 1, 2, 3, 4, the symmetric
\[
\{1, 2, 3\} \text{ into } \left(\frac{3}{2}\right) \text{ symmetric chains}
\]

sizes

\[
\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}
\]

Claim: one can always find such \(\left(\frac{n}{2}\right)\) symmetric chains that partition a set with \(n\) elements.

\(n=3\). Now add 4 to each chain element.

There are \(\binom{4}{2}\) but they are not symmetric.

But one can easily make them all symmetric (arrows)

Now are all symmetric.

Use induction on \(n\) to complete the proof.
Unit Distance Problem

Let there be $n$ points in $\mathbb{R}^2$. How many pairs can be created with unit distance? How many pairs can be created maximally by placing $n$ points?

Is $2n$ pairs (by placing them on a grid) the maximal number? It is known that $\frac{4n}{3}$ is an upper limit. We'll show that there are at most $\frac{3n}{2}$ pairs.

Case 1: There is a point with $\geq \sqrt{n}$ other points at unit distance. Pick $\sqrt{n}$ of them, call this set $P'$. Claim: $P'$ can only have $2n$ unit distances in total. Remove $P'$ from the circle.
**Theorem (Dilworth)**

Given a partial order \((X, \leq)\),

1. If largest chain in \(X\) has size \(k\), then \(X\) can be partitioned into \(k\) chains \((\rightarrow \text{Erdös-Szekeres})\).
2. If largest anti-chain in \(X\) has size \(k\), then \(X\) can be partitioned into \(k\) chains \((\rightarrow \text{decomposition in } (\binom{k}{2})\text{ chains})\).

**Example:** A sequence of 10 integers contains an increasing subsequence; for example

\[
3 \ 5 \ 9 \ 7 \ 4 \ 1 \ 6 \ 8 \ 10 \ 2
\]

**Proposition:** In any sequence of \(n\) elements one can find an increasing or decreasing subsequence of length \(\frac{n}{2}\).

**Disproof:** That is false, since \(n=10\), \(8\ 9\ 10\ 5\ 6\ 7\ 2\ 3\ 4\ 1\)

\((1, 2, \ldots, \frac{n}{2}), (\frac{n}{2}+1, \ldots, n)\).

**Proposition:** In any sequence of \(n\) elements one can find an increasing or decreasing subsequence of length \(\sqrt{n}\).

**Proof:** \(a_1, a_2, \ldots, a_n\), \(a_i \in \mathbb{N}\)

Pick \(a_i\) and consider the largest increasing subsequence ending (and including) with \(a_i\).
For example: 3 5 9 7 4 1 6 8 10 2

Note that elements with the same upper number form a decreasing subsequence.

Case 1: There is an upper number $\geq \sqrt{n}$

Case 2: One upper number must be repeated at least $n/\sqrt{n} = \sqrt{n}$ times by the pigeon-hole principle.

Recall: When we compare every pair of a set, the set is totally ordered. A partial order $\leq$ must satisfy:

- Transitivity,
- Reflexivity, and
- Anti-symmetry.

A set of elements with partial order is called a chain if every pair of elements can be compared. The set is an anti-chain, no pair can be compared.

Proof: (Dilworth)

E.g. for a): Impose $a_i \leq a_j$ iff $i < j$ and $a_i \leq a_j$
on the sequence problem. Then a chain is an increasing subsequence and an anti-chain a decreasing subsequence. The proposition follows.

E.g. for a): $X = \{1, \ldots, n\}$, partial order $(2^X, \subseteq)$

- largest anti-chain has size $\left(\begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array}\right)$.
- $2^X$ has a decomposition into symmetric chains.
a) Let \( X = \{x_1, \ldots, x_n\} \) with partial order \( \preceq \). For each \( x_i \), consider the length of the longest chain ending at \( x_i \):
\[
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow
\]
\( x_1, x_2, x_3, \ldots, x_n \)

Claim: All elements with the same upper-number form an anti-chain.
(Since if two had the same label one could extend one to form a longer chain).

b) By induction.

Assume the largest anti-chain in \( X \) has length \( k \).

To show: \( X \)
\[
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}
\]
\( k \)

induction on \( n \)

\[ b \]

Otherwise: There must be an anti-chain of length \( k \).

\( \tilde{C} \)
\[ \times \quad \& \quad \ldots \quad \times \]

Now each chain element of \( \tilde{C} \) is comparable to one of \( \tilde{C} \).
So one can divide the elements of $C$ into two categories. If

\[ t \in A \]

Transitivity forbids that $t$, a member of $C$, can be greater than one $x_i$ and less than another $x_j$ of $C$.

Problem 1: $A$ or $B$ could be empty. Impossible since else $C$ would not have been the longest chain.

Problem 2: An element $t \in C$ cannot be in $A$ and $B$ because of transitivity.
Probabilistic Method

Code: Alon, Spencer, title 1

At the beginning the probabilistic method seems to be merely a language to describe counting.

A discrete, such as $\Omega = \{1, ..., 6\}$ for tossing a die, is finite. Also $\sum p(i) = 1$ and $0 \leq p(i) \leq 1 \quad \forall i$. An event would be for example $A = \{\text{die comes up even}\} = \{2, 4, 6\}$. $P(A) = \frac{1}{2} = P(B)$ where $B = \{1, 3, 5\}$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

An example for a random function would here be $f(i) = i^2$, $i$ being the number of eyes of the die.

$E[f] = \sum p(i) f(i)$, a linear operation:

$E[αg(x) + βf(x)] = αE[g(x)] + βE[f(x)]$

Thm. (Erdős–Ko–Rado)

If $F$ is a family of $k$-element subsets of $S = \{1, ..., n\}$

s.t. $\forall S_i, S_j \in F \quad S_i \cap S_j \neq \emptyset$, then

$|F| \leq \binom{n-1}{k-1}$.
Recall: In the double-counting proof we considered \( A_i \).

**Proof:**

For \( 0 \leq i \leq n-1 \), let \( A_i = \{ i, i+1, \ldots, i+k-1 \} \mod n \) and \( |A_i| = k \).

Pick randomly and uniformly a permutation \( \sigma : \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\} \)
and pick randomly and uniformly an integer \( i \in \{0,1,\ldots,n-1\} \).

Consider \( A \) (underlined as it is a random set, depending on the two random variables, \( i \) and \( \sigma \)),
\[
\bar{A} = \{ \sigma(i), \sigma(i+1), \ldots, \sigma(i+k-1) \} \subseteq S
\]
where \( |\bar{A}| = k \).

Claim: \# different \( A_i \)'s that belong to \( S \) is \( \leq k \).

"Conditioned on \( \sigma \)" since \( \sigma \) is random.

For \( k = 1 \),

\[
\Pr[\bar{A} \in S | \sigma] \leq \frac{1}{n}
\]
for any \( \sigma \).

Since having chosen a starting point limits the number of consecutive sets,

Recall: Conditional probability

\[
\Pr[A | B] = \frac{\Pr[A \cap B]}{\Pr[B]}
\]
Because the probability that \( A = \) any given \( k \)-element subset of \( S \) is the same,

\[
\Pr[A \in \mathcal{F}] = \frac{\binom{n}{k}}{\binom{n}{k}}
\]

as for each given \( k \)-subset (of which there are \( \binom{n}{k} \) ),
the probability is \( \frac{1}{\binom{n}{k}} \) to be chosen.

Comparing this with \((\ast)\) gives \( |
\mathcal{F}| \leq \frac{k}{n} \cdot \binom{n}{k} \).  \( \square \)

**Proposition:** Let \( E_1, \ldots, E_k \) be disjoint events, \( \sum \Pr(E_i) = 1 \),
then if \( \Pr(A \cap E_i) \leq c \implies \Pr(A) \leq c \).

**Proof:**
\[
\Pr[A] = \sum_{i=1}^{k} \Pr(A \cap E_i) = \sum_{i=1}^{k} \Pr(A \cap E_i) \Pr(E_i)
\]
\[
\leq c \sum_{i=1}^{k} \Pr(E_i) = c \cdot c \cdot \square
\]

All of the double-counting proofs can be translated into probabilistic language.

**Def.: Tournament:** Any two of \( n \) players play a match where there is no draw. If \( i \) beats \( j \), a directed edge is drawn:

\[
\begin{align*}
& i \rightarrow j \\
& (i,j) \\
& i \rightarrow j
\end{align*}
\]

i.e. a directed complete graph.
Schütte's problem: Does there exist a tournament (knite $n$) such that $\forall A \subseteq V(T), |A| = k$, there is a $v_A \in V(T)$ s.t. $\forall u \in E(T) \forall u \in A$, i.e., for any $k$ players there is one who beats all of them. (Property $S_k$).

Remark: A 2-3 an infinite amount of players, $k$ one can always find a player who beats all of the others.

Proposition: (Erdős 1963) Yes, Schütte's proposition holds.

Thm.: $\forall n > k^2 2^{k^2/n^2} + O(1)$, $\exists$ a tournament with $n$ vertices that has property $S_k$.

Proof: $\forall A \subseteq S$ with $|A| = k$, $|S| = n$, define a random variable $X_A$. For each pair of vertices, let the outcome of their match have probability $\frac{1}{2}$. So there are $\binom{n}{2}$ ways to draw the tournament, each is equally likely. So our probability space consists of $2^{\binom{n}{2}}$ directed complete graphs on $n$ vertices.

$X_A = \begin{cases} 1 & \text{if } A \exists V_a \text{ that beats all elements of } A \\ 0 & \text{else} \end{cases}$

$X_A$ is called indicator. Let $\chi := \sum_{A \subseteq S, \#bad sets A}$
\[ E(X) = \sum A \Pr (X_A = 1) + 0 \Pr (X_A = 0) \]
\[ = \sum A \Pr (X_A = 1) = \binom{n}{k} \left( 1 - \frac{1}{2^k} \right)^{n-k} \text{, since:} \]
\[ \Pr (X_A = 1) = \Pr (A \text{ is } \text{bad}, \text{ no } v_A \text{ that beats all other}) \]

Look at a given set \( A \): As the probability that there is one to beat all others is \( 1/2^k \), the probability for a given vertex to be bad is \( 1 - \frac{1}{2^k} \). So:
\[ \prod_{v \in S \setminus A} \Pr (v \text{ is } \text{bad} \text{ for } A) = \left( 1 - \frac{1}{2^k} \right)^{n-k} \]

Applying Stirling:
\[ \binom{n}{k} \left( 1 - \frac{1}{2^k} \right)^{n-k} \leq \frac{(en)^k}{k^k} e^{-\frac{n-k}{2^k}} < 1 \]
\[ \frac{1}{1 - \frac{1}{2^k}} \leq e^{\frac{1}{2^k}} \]

If the average number of a random variable is less than 1, then there is at least one random variable \( X \) that is \( \leq E(X) \), so one \( X = 0 \)
\[ \Rightarrow T \text{ has property } S_k \quad \square \]

Alternatively:
\[
\Pr[(A_1 \text{ is bad}) \cup (A_2 \text{ is bad}) \cup \ldots \cup (A_k \text{ is bad})] \leq \sum_{i=1}^{\binom{n}{k}} \Pr(A_i \text{ is bad}) \\
= \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1
\]

\Rightarrow \Pr(\text{every } A_i \text{ is good}) > 0, \text{ so at least in one random graph, } S_k \text{ must hold.}

So we proved the proposition starting from a random tournament. We have no idea how the tournament looks like, i.e. a purely existential proof. So when you randomly construct graphs satisfy a property but delivering a concrete example is far more difficult.
More examples of the probabilistic method:

Problem 1: Finding a large cut in a graph.

Def.: A cut in a graph is a partition of $V(G)$ into two sets. The size of a cut is the number of edges going across the two sets.

Fig.: There are $2^{n-1}$ cuts possible on a graph with $n$ vertices. Or consider a complete graph:

$$\binom{n}{2} \quad \binom{n}{2}$$

$\#$ edges = size of cut

$$\Rightarrow \left(\frac{n}{2}\right)^2 \text{ out of } \frac{n^2}{2} - \frac{n}{2}$$

Thus: Any graph on $n$ vertices and $m$ edges has a cut of size at least $\frac{m}{2}$.

Proof: Key idea: Let's show that there are many cuts that satisfy the statement. Why could that be true and could even be easier than showing that there is one such cut? What's the average size of a cut? If this is large, at least one must be larger than it.

Average size of a cut: $\frac{\sum (\text{size of cut } i)}{2^{n-1}}$
Double counting: Rather than counting cut by cut, count edge by edge:

$$\sum_{i \text{ all cuts}} \text{(size of cut } i\text{)} \cdot \sum_{\text{all edges } E} \left( \text{# cuts which it lies}\right) = 2^{n-1} \cdot \sum_{\text{all edges } E} 2^{n-2} = \sum_{\text{all edges } E} 2^{n-1} = 2^{n-1}.$$ 

So the average is \( \frac{m}{2} \) \( \Rightarrow \) at least one cut must be larger than \( \frac{m}{2} \).

Alternatively: Probabilistic method.

Let \( X \) := size of a randomly picked cut.

\[
E(X) = \sum_{E} E(X_e) = \sum_{E} \Pr(X_e = 1), \quad \text{where} \quad E(X_e) = \begin{cases} 1 & \text{edge } e \text{ size of cut } X \\ 0 & \text{edge } e \text{ size of cut } \bar{X} \end{cases}
\]

So \( \sum \frac{1}{2} = \frac{m}{2} = E(X) \). So again there must be some cut whose size is at least \( \frac{m}{2} \). \( \Box \)
Problem 2: \( \varepsilon \)-nets

Let the points be people living in different regions.

Goal: Pick the minimum number of people such that from each region we have someone from each region. A representative.

Thus, let \( S_1, \ldots, S_m \) be subsets of \( X \), \( |X| = n \), and let \( X' \) be the smallest subset of \( X \) s.t. \( X' \cap S_i \neq \emptyset \) \( \forall i \). Assume further that \( |S_i| \geq \frac{n}{k} \) for some \( k \) and all \( i \).

\[ \exists X' \text{ s.t. } |X'| \leq 2k \ln m. \]

Proof: First via double-counting then probabilistic.

There are \( \binom{n}{t} \) subsets of \( X \) of size \( t = 2k \ln m \).

Let \( C \) count the subsets that do not work. To show:

\[ \sum_i \left( \text{# subsets without an element of } S_i \right) = \sum_i \binom{n-|S_i|}{t} < \sum_i \left( \frac{n}{k} \right) \leq m \left( \frac{n}{k} \right)! \leq \binom{n}{t}. \]

This inequality holds for \( t = 2k \ln m \). (Proof omitted.)
Alternatively: Probabilistic proof.
We want to pick a random set $X'$ and want to show that it fails with probability less than one.
Pick a random set of size $t$ where repetition of the elements is possible, i.e. $|X'| = t$ which we want to pick such that it fails: $\Pr(X' \text{ fails})$.

Let $\mathcal{E}_i := \text{event that set } S_i \text{ is not "hit" by } X'$, i.e. $S_i \cap X' = \emptyset$. So union bound

$$
\Pr(X' \text{ fails}) = \Pr\left( \bigcup_{i=1}^{m} \mathcal{E}_i \right) \leq \sum_{i=1}^{m} \Pr(\mathcal{E}_i) \quad \text{where}
$$

$$
\Pr(\mathcal{E}_i) = \left(1 - \frac{|S_i|}{n}\right)^t. \quad \text{Now let's use another inequality:}
$$

$$
\sum_{i=1}^{m} \left(1 - \frac{|S_i|}{n}\right)^t \leq \sum_{i=1}^{m} \left(1 - \frac{n}{k}\right)^t = \sum_{i=1}^{m} \left(1 - \frac{1}{k}\right)^t = m \left(1 - \frac{1}{k}\right)^t
$$

$$
m \left(1 - \frac{1}{k}\right)^t \leq m e^{-\frac{t}{k}} = \frac{m}{e^{\frac{t}{k}}} \quad \left(1 - \frac{1}{k} < e^{-\frac{1}{k}} \right)
$$

$$
= m e^{-\frac{t}{k}} < 1. \quad \text{So } \Pr(X' \text{ fails}) < 1
$$

$\Rightarrow$ there must be an $X'$ satisfying the claim. $\square$

Alternatively, pigeon-hole principle:
Assume an element is in $r$ sets. So

$$
m \cdot \frac{n}{k} \leq nr
$$
Recall the $\varepsilon$-net problem of last time:

$X = \{1, \ldots, n\}$, $S_i \subseteq X$ s.t. $i \in \{1, \ldots, m\}$ and $|S_i| \leq \frac{n}{k}$ for all $i$. $\exists \tilde{X} \subseteq X$ s.t. $\tilde{X} \cap S_i \neq \emptyset$ for all $i$; and $|\tilde{X}| = 2k \ln m$.

We saw three ways of proving this: combinatorial, probabilistic, and the "greedy" method from exercise sheet 8. The heart of the probabilistic proof was:

Randomly pick $t$ elements:

$$Pr(\text{at least one set not hit}) \leq m \left(1 - \frac{1}{k}\right)^t$$

Let's generalize the problem a bit:

Require $\tilde{X} \cap S_i = "\text{at least half the elements of } S_i."

Pick a random sample with choosing an element with probability $\frac{1}{2}$.

Let us allow some error first:

$$Pr(\text{at least } \frac{1}{2} |S_i| \text{ elements picked from } S_i \text{ is not easy},$$

as it is a sum of binomials.

Today we will look at tools to evaluate this, namely

inequality of Chebychev and inequality of Markov.
let $X = \sum_{i} x_i$ where $x_i = \begin{cases} 1 & \text{if } i^{th} \text{ element is picked} \\ 0 & \text{else} \end{cases}$

$E(X) = \frac{|S_i|}{2}$. What is $Pr(X \text{ is far from } E(X))$?

Say $Pr(X \geq kE(X))$.

E.g.: let there be $n$ elements and pick each with probability $\frac{1}{2}$ and define $x_i = 1$ if $i^{th}$ is picked, zero otherwise.

# elements picked = $X = \sum_{i=1}^{n} x_i$

Recall: $E(X) = \sum_{X} X \cdot Pr(X)$.

In a combinatorial setting, consider the unweighted average of $n$ numbers

$A = \frac{\sum a_i}{n}$. So the equivalent question for $(X)$ would be: how many $a_i$'s are $\geq k \cdot A$?

Pick $x$ of the $a_i$'s such that $x \cdot \frac{a}{x}$ is as big as possible.

$A = \frac{\sum_{X} a_i + (n-x) \cdot 0}{n}$

$x = \frac{n}{k}$

numbers here

numbers here
\[ A = \frac{1}{n} \sum_{i=1}^{n} a_i = \frac{1}{n} \left( \sum_{a_i < kA} a_i + \sum_{a_i \geq kA} a_i \right) \]

\[ \geq \frac{1}{n} \sum_{a_i \geq kA} a_i \geq \frac{1}{n} kA \sum_{a_i \geq kA} 1 \]

\[ x \leq \frac{n}{k} \quad \text{Hence} \quad A \geq \frac{k}{n} \cdot A \cdot x. \]

One can do the same calculation with \( E(X) \) instead of \( A \) to get Markov's inequality:

\[ \Pr \left( X \geq k \cdot E(X) \right) \leq \frac{1}{k} \]

So let's do it: Let \( \Pr(X) := \Pr(X = x) \)

\[ E(X) = \sum_{X < kE(X)} x \Pr(X) + \sum_{X \geq kE(X)} x \Pr(X) \]

\[ \geq \sum_{X \geq kE(X)} kE(X) \Pr(X) = kE(X) \sum_{X \geq kE(X)} \Pr(X) \]

So

\[ E(X) \geq kE(X) \Pr(X \geq kE(X)) \]

\[ \Rightarrow \Pr(X \geq kE(X)) \leq \frac{1}{k} \]

Note: this is only true for \( X \geq 0 \) \( \forall X \).
let's return to the initial setting.

\[ \Pr(X \geq n) = \frac{1}{2^n} \], but Markov says

\[ \Pr(X \geq 2, \frac{n}{2} = 2 \cdot E(X)) \leq \frac{1}{2} \], i.e. useless here.

Only two assumptions were important for Markov's inequality: \( X \geq 0 \) \( \forall X \) and \( \int E(X) \).

In order to improve Markov's inequality we must introduce the variance. For example in the following two examples we have the same result of Markov:

\[ \Pr(9) = \Pr(10) = \frac{1}{2} \]

\[ \Pr(0) = \Pr(20) = \frac{1}{2} \]

\[ \begin{array}{cccc}
9 & 10 & 11 & \\text{o}^2 = 100 \\
0 & 10 & 20 & \text{o}^2 = 100
\end{array} \]

i.e. some measure of how far the random variables are away from the mean, \( E(X) \), needs to be introduced:

\( \Rightarrow \) variance.

\[ \text{Var } X := E((X - E(X))^2) = \sigma^2 \]

E.g.:

\[ x_i = \begin{cases} 1 & \text{with prob. } \frac{1}{2} \end{cases} \quad \text{Var } X_i = \frac{1}{4} \]

\[ X = \sum_{i=0}^{n} x_i \Rightarrow E(X) = \frac{n}{2} \], so

\[ \text{Var } X = \left< (X - \frac{n}{2})^2 \right> = \sum_{i=0}^{n} (\frac{n}{2} - i)^2 \left( \frac{1}{2^n} \right) = \]
$$= \frac{1}{2^n} \left( \sum_{i=0}^{n} \left( \binom{n}{i}^2 (\frac{n}{2})^i - 2 i \binom{n}{i} \frac{n}{2} + i^2 \binom{n}{i} \right) \right) = \frac{n}{4}.$$ 

Note that \( \sum_{i=0}^{n} i \binom{n}{i} = n 2^{n-1} \) like picking a team of site \( i \) and a captain from it.

And \( \sum_{i=0}^{n} i^2 \binom{n}{i} = n 2^{n-1} (n(n+1)) 2^{n-2} \) and \( \sum_{i=0}^{n} \binom{n}{i} = 2^n. \)

\[ \Rightarrow \text{Var } X = \frac{n}{4}. \quad \text{Let } \text{Var } X := V[X]. \]

**Proposition:** \( V[X + Y] = V[X] + V[Y] \), if \( X, Y \) independent.

**Proof:** By definition: \( V[(X+Y)] = E[(X+Y - E[X] - E[Y])] \)

= \( V[X] + V[Y] - E[2(X-E(X))(Y-E(Y))] \).

If \( X \) and \( Y \) are independent, then

\( E(X,Y) = E(X) \cdot E(Y) . \quad \Box \)

Let's motivate Chebychev's inequality by the following: \( \Pr \left( (X-E(X))^2 \geq t \right) \) on which we apply Markov's inequality.
\[ Pr \left( \left( X - E(X) \right)^2 \geq t \right) \leq \frac{\mathbb{V}[X]}{t} \] by Markov.

So \[ Pr \left( |X - E(X)| \geq t \right) \leq \frac{\mathbb{V}[X]}{t^2}, \] which is Chebychev's inequality. \( \square \)

Let us revisit \[ Pr \left( X \geq n \right) \leq Pr \left( |X - \frac{n}{2}| \geq \frac{n}{2} \right) \]

\[ \leq \frac{n}{n^2} = \frac{1}{n}. \] Still far from \( \frac{1}{2^n} \) but better than \( \frac{1}{2}. \)

E.g.: let's consider the central binomial coefficient \( \binom{n}{n/2} \).

As \[ \sum_{i=0}^{n} \binom{n}{i} = 2^n, \quad \binom{n}{n/2} = \frac{2^n}{\sqrt{n+1}}. \]

We can get a much better bound, by the following argument: let there be \( n \) bits, placing 0 or 1 by probability 1/2. Let \( X = \# \text{ ones} = \sum_{i=1}^{n} X_i \)

\[ X = \sum_{i} X_i \Rightarrow Pr \left( \# \text{ ones are } \frac{n}{2} \pm \frac{\sqrt{n}}{2} \right) \]

\[ \leq Pr \left( \left| X - \frac{n}{2} \right| \geq \frac{\sqrt{n}}{2} \right) \leq \frac{n}{n^2} = \frac{1}{2^n}. \]

\[ \sum_{i=\frac{n}{2} - \frac{\sqrt{n}}{2}}^{\frac{n}{2} + \frac{\sqrt{n}}{2}} \binom{n}{i} \leq 2 \sum_{i=\frac{n}{2} - \frac{\sqrt{n}}{2}}^{\frac{n}{2} + \frac{\sqrt{n}}{2}} \binom{n}{i} = 2^{n - 2} \binom{n}{\frac{n}{2}} \]

so \( \left( \frac{n}{2} \right) \geq \frac{2^n}{n^{\frac{3}{2}} \cdot e^{\gamma n}}. \)

Proofs are exam relevant.
Chébyshev's Inequality

For the random variable \( X \), \( E(X) = \mu \), \( \text{Var}(X) = \sigma^2 \), then

\[
\Pr \left( |X - \mu| \geq 2\sigma \right) \leq \frac{1}{4}.
\]

Eq.

Def.: Given a set \( \{x_1, x_2, \ldots, x_k\} \) is said to be a distinct sum set (DS sum set) if every subset has a distinct sum, i.e., for \( S_1, S_2 \subseteq \{x_1, \ldots, x_k\} \)

\[
S_1 \neq S_2 \implies \sum_{x \in S_1} x \neq \sum_{y \in S_2} y.
\]

E.g.: \( \{1, 2\} \) is a DS set.

Problem: Given \( \{1, 2, \ldots, n\} \), what is the size of the largest subset that is a distinct sum set?

Clearly, \( \{2^1, 2^2, \ldots, 2^n\} \subseteq \{1, 2, \ldots, n\} \) as each integer in this subset has a unique binary representation. Hence

\[
\log_2 n \leq \text{"size of largest DS set"} := |\text{DS}|
\]

Assume \( \{x_1, x_2, \ldots, x_k\} = \text{DS} \). Now

\[
k < 2^k = \# \text{ of distinct sums} \leq nk
\]

by def.

of DS set

\[
\implies k \geq \log_2 nk = \log_2 n + \log_2 k
\]
$$k \leq \log_2 n + \log\log n + 1$$ is thus an upper bound of $|DS|$:

$$1 \leq |DS| \leq \log_2 n + \log(\log_2 n) + 1$$

Thm.: The largest DS subset of $\{1, \ldots, n\}$ has size at most $\log_2 n + \frac{1}{2} \log_2 \log_2 n + C$.

Proof: Suppose $\{x_1, x_2, \ldots, x_k\} = DS$, i.e., the largest DS subset of $\{1, \ldots, n\}$. Take a random sample of DS by picking each element with prob. $\frac{1}{2}$, and then consider the sum.

Let $E_i = \begin{cases} 1 & \text{if } x_i \text{ is picked} \\ 0 & \text{else} \end{cases}$

$E(E_i) = \frac{1}{2}$ and $\text{Var } E_i = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Let $X = \sum_{i=1}^{k} E_i X_i$, i.e., the sum of the random subset.

$$E(X) = \sum_{i=1}^{k} E(E_i X_i) = \sum_{i=1}^{k} E_i \cdot E(X_i) = \frac{1}{2} \sum_{i=1}^{k} X_i.$$  

$$\text{Var } X = \sum_{i=1}^{k} \text{Var } (E_i X_i) = \sum_{i=1}^{k} E_i^2 \cdot \text{Var } E_i = \frac{1}{4} \sum_{i=1}^{k} X_i^2.$$  

By Chebyshev: $\Pr(|X - \mu| > 2\sigma) < \frac{4}{\sigma^2}$.  

The variance can easily be bounded above by \( \text{Var} X \leq \frac{n^{3/2} k}{4} \), so \( \sigma \leq \frac{n^{3/2} k}{2} \). For example, consider the following:

\[
\Pr \left( \left| X - \mu \right| > 2 \cdot \frac{n^{3/2} k}{2} \right) < \frac{1}{2^2}
\]

Hence,

\[
\Rightarrow \Pr \left( \left| X - \mu \right| \leq 2 \cdot \frac{n^{3/2} k}{2} \right) \geq 1 - \frac{1}{2^2}.
\]

\[
\{ x_1, x_2, \ldots, x_k \}
\]

\[
\downarrow \text{random sample and add terms; there must be } 2^k \text{ distinct sums, by def. of DS,}
\]

\[
\Pr \left( X \text{ is in the shaded region} \right) \leq \frac{2n^{3/2} k}{2^k}
\]

\[
\Pr \left( \left| X - \mu \right| < \frac{2n^{3/2} k}{2} \right).
\]

So,

\[
1 - \frac{1}{2^2} \leq \Pr \left( \left| X - \mu \right| < \frac{2n^{3/2} k}{2} \right) \leq \frac{2n^{3/2} k}{2^k}
\]

Pick a positive \( \varepsilon \), say \( \varepsilon = \frac{1}{3} \) to get

\[
2^k \left( 1 - \frac{1}{3} \right) \leq \frac{2}{3} \cdot \frac{n^{3/2} k}{2^k} \Rightarrow 2^k \leq \frac{3\sqrt{n} \cdot n^{-1} k}{2}.
\]
Hence \( k \leq \log_2 n + \log_2 \sqrt{k} + \log_2 \frac{3 + \sqrt{3}}{2} \leq \log_2 n + \frac{\log_2 \log_2 n}{2} + \epsilon \)

More probabilistic methods:

Def.: \( G(V, E) \) is a planar graph if it can be drawn in the plane without crossing edges. \( G \) is planar iff \( G \) does not have a \( K_5 \) or \( K_{3,3} \) topological minor.

Thm.: If \( G \) is planar and \( V(G) = n \), then
\[
|E(G)| \leq 3n - 6.
\]

Proof: \( \sum_{f \in \mathcal{F}(G)} d(f) = 2|E(G)| \geq \sum_{f \in \mathcal{F}(G)} 3 = 3f \), i.e.

\( 2m \geq 3f \) and by Euler, \( n - m + f = 2 \), so \( m \leq 3n - 6 \).

Def.: The crossing number of \( G \) is the minimum number of crossing edges over all drawings of \( G \), (i.e. the "best" drawing of \( G \)). Let the crossing number of \( G \) be \( \text{cr}(G) \).

E.g.: \( \text{cr}(K_4) = 0 \), \( \text{cr}(K_5) = 1 \).
Lemma: (Crossing Lemma) If $G$ is a $n$-vertex graph with $e \geq 4n$, then $(|E(G)| = e)$

$$cr(G) \geq \frac{e^3}{64n^2}.$$ 

Proof: Clearly, $cr(G) \geq e - (3n - 6) = e - 3n + 6 \geq e - 3n$

So $cr(G) \geq e - 3n$ for any graph.

Let us pick each vertex of $G$ with probability $p$, and require that both vertices of an edge must be picked for the edge to survive.

Let $X := cr(G') - e' + 3n' \geq 0$ where $G'$ is picked from the given graph. So it is a random variable.

$$E(X) = E(cr(G')) - E(e') + 3 E(n') \geq 0.$$ 

Let the picked $G'$ be drawn with minimal crossings. A given crossing in $G'$ survives if 4 vertices are chosen, hence $E(cr(G')) = cr(G) \cdot p^4$. Thus

$$E(X) = cr(G) \cdot p^4 - ep^2 + 3np \geq 0$$

$$\Rightarrow cr(G) \geq ep^2 - 3np^3 = f(p)$$ and maximize $f(p)$, so $f'(p) = 0 \Rightarrow p = \frac{4n}{e} \leq 1$ so a valid probability. Plugging this in gives: $cr(G) \geq \frac{e^3}{64n^2}$. $\square$
Problem: Let there be \( n \) lines and \( n \) points in \( \mathbb{R}^2 \). What is the maximum number of incidences between points and lines? E.g.,

Two incidences:
\[(P_2, l_3), (P_2, l_1)\].

Thus, given \( n \) points and \( n \) points,

\[
\text{max \# incidences} \leq C \cdot (n^4/n + n)
\]

Proof: Define a graph \( G \) as follows. Let \( V(G) \) be the \( n \) points and \( E(G) \) be two consecutive points on a line. E.g.,

which we want to bound from above,

\[
\# incidences \leq |E(G)| + \frac{\# lines}{n} = |E(G)| + n.
\]

By the crossing lemma,

\[
\text{cr}(G) \geq \frac{|E(G)|^3}{64 \cdot n^2}
\]

A crossing can happen when 2 lines cross, so the maximum number of crossings is \( \binom{n}{2} \), so

\[
\frac{|E(G)|^3}{64 \cdot n^2} \leq \text{cr}(G) \leq \binom{n}{2} \leq \frac{n^2}{2}.
\]
\[ |E(G)|^3 \leq 32 n^4 \implies |E(G)| \leq \frac{\sqrt[3]{32}}{2} n^{9/3} \]

\[ \# \text{ incidence} \leq \frac{\sqrt[3]{32}}{2} n^{9/3} + n \]
Linear Algebra Method

Introductory problems:

1) Let there be \( n \) people in a city who form clubs according to two rules:
   
a) \( |A_i| = \text{even} \ \forall \ i \)
   
b) \( |A_i \cap A_j| = \text{even} \ \forall \ i \neq j \)

What is the max \# sets \( A_i \) that satisfy these rules? 

E.g., \( \{1,2,3,4\} \) clearly satisfies the conditions. So at least \( \frac{n}{2} \) sets can be chosen.

See each point couple as one point to see that there are at least \( 2^{n/2} \) clubs.

Hw: Is this the best possible lower bound?

2) New rules:
   
a') \( |A_i| = \text{odd} \ \forall \ i \) 
   
b') \( |A_i \cap A_j| = \text{even} \ \forall \ i \neq j \)

Same question: What is the maximum number of \( A_i \) satisfying those rules?

Select a special point and include it in each club, but here rule b) doesn't work.
Thus: \((n)\) max \# sets is \(n\)

**Proof:**

\[
\begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_m \\
\end{pmatrix} = 
\begin{pmatrix}
X_1 & X_2 & \ldots & X_n \\
0 & 1 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

Consider the following incidence matrix.

If \(X_i\) is in \(A_j\), then the \(ij\) entry is one, else zero.

Call this matrix \((a_{ij})\).

For the sake of contradiction, assume \(m > n\).

**Claim:** One can assign constant \(X_i\) to \(A_j\) such that, \(X_i \in \{0, 1\}\), letting 0 represent even and 1 represent odd, such that

\[
\sum_{i=1}^{m} X_i \cdot V_i = (0, 0, \ldots, 0),
\]

where \(V_i\) is the \(i\)th row vector, and not all \(X_i\)'s are zero.

(\# choices for \((X_1, \ldots, X_m)\) = \(2^m\) and for each choice \(\sum_{i=1}^{m} X_i \cdot V_i = (\mu_1, \mu_2, \ldots, \mu_m) \in \{0, 1\}^n\)

where each \(\mu_i\) is either odd or even \((1\ or\ 0)\).

If \(2^m > 2^n\) then there will be two choices \((X_1', \ldots, X_m')\) and \((X_1'', \ldots, X_m'')\) such that

\[
\sum_{i=1}^{m} X_i' \cdot V_i \equiv \sum_{i=1}^{m} X_i'' \cdot V_i \pmod{2}
\]

\[
\sum_{i=1}^{m} (X_i' - X_i'') \cdot V_i \equiv 0 \pmod{2}.
\]

End of claim.
One can also see this using this fact from linear algebra.
Any set of linearly independent vectors in an \( n \)-dimensional vector space has at most \( n \) elements.
Let \( \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m \) be vectors such that for each \( i \),
\[
\sum_{i=1}^{m} \mathbf{z}_i \cdot \mathbf{v}_i = 0 \quad \text{(mod 2)},
\]
where \( \mathbf{v}_i \) is a basis vector. Taking the dot product with \( \mathbf{v}_i \),
\[
\sum_{i=1}^{m} \mathbf{z}_i \cdot \mathbf{v}_i^2 = 0 \quad \text{(mod 2)}
\]
even by rule b) even by rule b)
\[
\mathbf{z}_1 \cdot \mathbf{v}_1 \mathbf{v}_1 + \mathbf{z}_2 \cdot \mathbf{v}_2 \mathbf{v}_2 + \cdots + \mathbf{z}_m \cdot \mathbf{v}_m \mathbf{v}_m = 0
\]
odd by rule a')

\[
\Rightarrow \quad \mathbf{z}_i = 0
\]

Next multiply (dot product) with \( \mathbf{v}_2 = 0 \). This can be done with all \( \mathbf{v}_i \) thus \( \mathbf{z}_i = 0 \forall i \). But this contradicts our assumption. \( \ast \)

**Thm.** (Fisher's Inequality)

Let \( A_1, \ldots, A_m \subseteq X \), \( |X| = n \) and \( |A_i \cap A_j| = k \forall i \neq j \).
Then \( m \leq n \).
E.g.: i) $k = 1$, $m = n$

ii) Fano Plane, 7 pts, 7 sets $A_i$ ("lines"), s.t. any 2 pts can be connected by precisely one line and any 2 lines have precisely one pt in common.

$|A_i| = 3$, $m = n = 7$

**Proof:** Again an incidence matrix: $A$ in row vectors $v_i$; $x_1, x_2, \ldots, x_n$,

$A_i = v_i (0, 1, 1, \ldots, 1) \in \{0, 1\}^n$, so $\langle v_i, v_i \rangle = |A_i|$

For the sake of contradiction, assume that $m > n$, then

$\exists z_i, i \in \{1, \ldots, m\}$, not all zero, such that

$\sum_{i=1}^{m} z_i v_i = 0 \in \mathbb{R}^n$, i.e. there must be a non-trivial linear dependence. Now consider

$\langle \sum_{i=1}^{m} z_i v_i, \sum_{j=1}^{m} z_j v_j \rangle$

$= \langle 0, 0 \rangle = \sum_{i=1}^{m} z_i^2 \langle v_i, v_i \rangle + \sum_{i < j} z_i z_j \langle v_i, v_j \rangle = \frac{1}{|A_i|} 1_{|A_i|}$

**Remark:** $|A_i| > k \forall i$, except perhaps 1.
\[ \sum_{i=1}^{m} a_i^2 (|A_i| - k) + \sum_{i \neq j} a_i a_j k + \sum_{i=1}^{m} a_i^2 k = \]

\[ = \sum_{i=1}^{m} a_i^2 (|A_i| - k) + k \cdot \left( \sum_{i=1}^{m} a_i \sum_{j=1}^{m} a_j \right) = 0 \]

\[ > 0 \quad \forall i \]

A sum of positive terms equals to zero if both summands are zero. This only holds if \( a_i = 0 \quad \forall i \). ✗
More about the linear algebra method.

Thm.: (Sauer, Shelah, Vapnik-Chervonenkis)

Let \( A_1, A_2, \ldots, A_m \subseteq X \) where \(|X| = n\).

Call \( S \subseteq X \) shattered if \( \forall T \subseteq S \exists A_i \text{ s.t. } A_i \cap S = T \). Then:

i) If there is a shattered set of \( k \) elements, then \( m \geq 2^k \).

ii) If we take all subsets of \( X \) that have at most \( k \) elements then there is no shattered set of size \( k+1 \).

If \( m > \sum_{i=0}^{k} \binom{n}{i} \) then there is a shattered set of size \( \geq k+1 \). This bound is tight (see ii)).

Recall: "Linear algebra method": In every \( n \)-dimensional vector space, the \( n \) linearly independent vectors is at most \( n \).

Proof:

Its contrapositive: If there is no shattered set of size \( k+1 \) then \( m \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \)

we cannot prove strict inequalities via the linear algebra method.
Let us define an incidence matrix:
\[
\begin{pmatrix}
N & N & \cdots & N \\
0 & 1 & \cdots & k \\
\emptyset & \{x_1\}, \{x_2\}, \ldots, \{x_k\}
\end{pmatrix}
\]
For each k-element subset of \(X\) there is a column.

\[A_i = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & \cdots \end{pmatrix} \]

Let \(\alpha_{ij} = \begin{cases} 1 & \text{if } B_j \subseteq A_i \\ 0 & \text{else} \end{cases}\)

Let us consider the column vectors now. The ones denote the \(A_i\) that contain a given \(B_j\).
Suppose for contradiction that \(m > \binom{n}{0} + \cdots + \binom{n}{k}\) i.e. more rows than columns. A row vector \(v_i\) is element of \(\{0,1\}^n\). The assumption implies \(\exists i \text{ not all zero, s.t. } \sum_{i=1}^{m} v_i = 0 \in \{0,1\}^n\), i.e. adding up each column (multiplied by \(v_i\) row-wise) must be zero. So for every at most \(k\)-element subset \(B_j \subseteq X\), \(\sum_{i=1}^{m} v_i = 0\). We can define this sum for any set, \(B_j \subseteq A_i\) so let \(B \subseteq X\) and define \(f(B) = \sum v_i\) consider a smallest set \(B \subseteq X\)

\[A_i \supseteq B\]

for which \(f(B) = \alpha \neq 0\). One such set is the \(A_i\) that contains the largest number of elements and for which \(\exists j \neq 0:\ f(A_j) = \sum v_i = \alpha_j \Rightarrow |A_j| = k + 1\).
Claim: $\forall C \subseteq B$, $f(C) = (-1)^{1 - |B \cap C|}$, $f(B) \neq 0$

This holds, since

$$f(C) := \sum_i x_i \quad \text{and note that } f(B) = f(B)$$

$$B \cap A_i = \emptyset$$

So, $f(C) = \sum_i x_i = 0 \Rightarrow \exists A_i \ni B = C \Rightarrow B \text{ is shrunk.}$

This follows from the inclusion-exclusion principle.

For instance, if $C = B \setminus \{b\}$, consider $A_i$, $A_i \cap B = C$.

$$f(C) = \sum_i x_i = \sum_i x_i - \sum_i x_i = \sum_i x_i$$

$$A_i \cap B = C \quad A_i \subseteq C \quad A_i \supseteq B$$

$$= f(C) - f(B) = f(C) - \alpha$$

$$= -\alpha \quad \text{since we defined } B \text{ as the smallest set for which } f(B) \neq 0,$$

thus $f(C) = 0$.

This can be generalised via induction: E.g. $f(C) = -\alpha + \alpha + \alpha$.
Application: $0$ people

$|A_i| \geq \varepsilon n \quad (i = 1, \ldots, m)$

Pick a set of people s.t. $A_i$ is "fairly" represented.

If there is no sheltered set of size $k+1$ then

$$C_k \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

representatives are sufficient

(Vapnik-Chervonenkis). Note that the number of representatives does not depend on $n$. So even in a big country one need not ask more people than in a small country. The selection of those representatives is more important than the total number of participants.

Thus: (Larman-Rodgers-Seidel)

Def.: $X \subseteq \mathbb{R}^d$ is said to be a $k$-distance set if there are at most $k$ distinct distances $|x-y|$ determined by $x \neq y \in X$.

E.g.: 1-distance set in $\mathbb{R}^2$:

at most 3 elements.

In $\mathbb{R}^d$, $|S| \leq d + 1$.

If $S \subseteq \mathbb{R}^d$ is a 2-distance set then

$$|S| \leq \frac{(d+1)(d+1)}{2} \quad \text("nearly tight")$$
Proof: (Linear algebra)

The linear space will be a space of polynomials, here.
Let the two distances of $s$ be denoted by $\alpha$ and $\beta$.

For all $t \in \mathbb{R}^d$, define a function on $t \in \mathbb{R}^d$

$$F_s(t) = (||t - s||^2 - \alpha^2)(||t - s||^2 - \beta^2)$$

If $t \neq s$ and $t \in S$ then $F_s(t) = 0$, and $F_s(s) = (\alpha\beta)^2 \neq 0$.

There exist no $\mathbb{R}_1, \ldots, \mathbb{R}_{15}$, not all zero, such that

$$\sum_{s \in S} F_s(t) = 0,$$

i.e. the functions $F_s(t)$ satisfy $s \not\in S$ are linearly independent. This can be seen by setting

$$t - s : \mathbb{R}_s \times \mathbb{R}_s = 0 \Rightarrow \mathbb{R}_s = 0 \ \forall \ s \in S \ \forall \ y.$$

It remains to prove that all functions $F_s(t)$, $s \in S$, belong to a low-dimensional vector space of polynomials;

$$||t - s||^2 = \langle t - s, t - s \rangle = ||t||^2 - 2\langle s, t \rangle + ||s||^2$$

So, as $||t||^2 = t_1^2 + \ldots + t_d^2$. Multiply out $F_s(t) = \sum_{i=1}^{d} (t_i^2 - 2s_i t_i + ||s||^2)$

$$= ||t||^2 - \sum_{i} s_i t_i + 8 \sum_{i < j} s_i s_j t_i t_j + 4 \sum_{i} s_i^2 t_i^2 + \to$$
Prop.: Let there be a family of sets $A_1, \ldots, A_m$, $|A_i| = k$ and $m < 2^{k-1}$. There exists a 2-coloring of $\bigcup_{i=1}^{m} A_i$ such that no $A_i$ is monochromatic.

Proof: (Erdős)

Consider a random coloring of the points, prob. $\frac{1}{2}$, we want to show: $\Pr\left( \exists \text{ monochromatic } A_i \right) > 0$.

$\Pr\left( \exists \text{ monochromatic } A_i \right) \leq \frac{m}{2^k} = \frac{m}{2^{k-1}} < 1$ by assumption $\Rightarrow \exists$ a 2-coloring.

Alternatively: Algorithmic proof.

Proof: (Erdős - Selfridge)

Greedy algorithm. (Consider two people alternatively, trying to have a monochromatic set by painting a point orange or green). Strategy for 2 players: choose the point that is "most important" and color it by orange or green. So how do we define most important?
The 1st player (he starts the game): \( A_1 \)

Each set \( A_i \) gets a weight \( w(A_i) \).

Originally \( w(A_i) = 1 \forall i \). If \( A_3 \)

1st player puts a point into \( A_i \), then \( w(A_i) = 0 \), since he no longer has to worry about the set since it can no longer be made monochromatic in green. The more green points in a set, the more dangerous it becomes for the 1st player, thus define a set with green points only as having weight \( w_1(A_i) = 2^{(# \text{green points in } A_i)} \)

The initial condition is:

\[
\sum_{i=1}^{m} w_1(A_i) = m < 2^{k-1}
\]

The weight of a point \( p \),

\[
w_1(p) = \sum_{A_i \ni p} w_1(A_i)
\]

since it is more important if it is in more sets.

Now again: Strategy for the players is such that at each turn the point with the highest weight \( w(p) \) is pointed.

2nd player (green): \( w_2(A_i) = 2^{(# \text{orange points in } A_i)} \)

\[
w_2(p) = \sum_{A_i \ni p} w_2(A_i)
\]

\( w_2(A_i) = 0 \) if \( A_i \) has a green pt.
Total weight for 2nd player (after 1st player made his first choice):

\[ \sum_{i=1}^{m} w_i(A_i) \leq 2m < 2^k \]

Essence of the proof: If both players follow the just described greedy strategy then their total weights can only decrease! So neither of the players can win!

Let us look at this from the point of view of the orange player: Green cannot win, since for that a set must have weight \( 2^k \), but this is impossible since

\[ \sum_{i=1}^{m} w_i(A_i) = m < 2^{k-1} \]

initially and only decreases. The same is true for the green player, since

\[ \sum_{i=1}^{m} w_i(A_i) \leq 2m < 2^k \]

Why is this true that the total weights can only decrease?

From point of view of orange player, consider the last point that orange picked: As it was the best point possible,

\[ w_4(p) = \sum_{A_i \in \partial} w_i(A_i) \]

was maximal, and thus the total weight for orange decreases by \( w_4(p) \) since \( w_i(A_i) = 0 \) if an orange point is in it.

But if green picks a point \( q \), the total weights of \( A_i \) increase by \( w(q) \). But the decrease is bigger than the
increase, since: $\omega(p) \geq \omega(q)$. A similar argument shows that the total weight for the green player only decreases. $\Box$

The idea of introducing "fighting players" to replace the probabilistic method is the most popular.

**EXAM:** Written, 5 or 6 questions,

- 1 proof of a theorem covered in class, send a list to (janospach@gmail.com)?
- At least one problem from a homework assignment (remember solutions)

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**2 Ramsey Theory**

**Thm.:** $\forall n \exists R(n)$ s.t. no matter how we color the edges of a complete graph with $\geq R(n)$ vertices by two colors, there exists a monochromatic complete subgraph $K_n$.

One can extend the problem by introducing more colors; then the minimal number of vertices such that a monochromatic complete graph is guaranteed will depend on $c$ as well: $R(n, c)$. 
For hypergraphs, take more vertices to define an edge:

- Triples of vertices: 3-uniform hypergraph
- Quadruples of vertices: 4-uniform hypergraph

All $\binom{R}{3}$ triples, i.e. a complete 3-uniform hypergraph on $R$ vertices.
Color each triple by one of $c$ given colors.

**Thm. (General Ramsey Theorem) 1930 (Hajnal Erdős Theorem)**

For every $n, c \in \mathbb{R}_3(n, c)$ such that no matter how we color all triples of an $\geq \binom{R}{3}(n, c)$-element set $X$ with $c$ colors, there is an $n$-element subset $Y \subseteq X$ such that all $\binom{Y}{3}$ triples induced by them are of the same color.

**Thm. (Erdős-Szekeres Convex $n$-gon theorem 1935)**

For every $n$, there exists $E(n)$ such that any set of $\geq E(n)$ points in the plane, no three of them collinear, contains the vertex set of a convex $n$-gon.

E.g.: (When spanning a rubber band around, all vertices in a convex graph are touched.)
with any 5 points in the plane, not three of them collinear, one can place a "rubber band" around them and the four touched vertices are in convex position. Problematic cases:

Pasch axiom: A line cannot intersect all three sides of a triangle unless it passes through a vertex.

I.e., E5-thm. says that if one has enough points, one can find a convex n-gon.

Proof: \[ E(n) \leq R_4(n, 2) = R_4, \quad n \geq 5, \]

if we color all 4-tuples of an R-element set by 2 colors then we find \( n \) elements such that all \( \binom{n}{4} \) 4-tuples induced by them are of the same color.
R points, each 4 element subset is colored in one of two colors.

different color, since the convex hull is a triangle.

By Ramsey's theorem.

By E. Klem's observation there are no 5 pts. such that all 4-tuples determined by them are green. So \( E \) n pts. such that all 4-tuples determined by them are orange. \( \Rightarrow \) These n points form the vertex set of a convex n-gon.

Triangulation \( \square \)